# EXAMINING EXPLORATIVE INSTRUCTION ACCORDING TO THE REALIZATION TREE ASSESSMENT TOOL 

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In this paper we present a comparison of 10 lessons based on the same task - the Hexagon task. We used the RTA (Realization Tree Assessment) tool in order to compare the implementation of this task by 10 middle school teachers undergoing professional development intended to enhance explorative instruction. We focused on three aspects in our comparison: the number of realizations, the links between the realizations, and the narratives of 'saming' algebraic expressions. Results show a wide variance between lessons in number of realizations and in the extent to which links were made between them. The quantification of these aspects enabled us to rank the lessons according to RTA "robustness" to provide a measure of explorative instruction and link it with grade level and track.

## INTRODUCTION

Focusing on mathematical concepts during the lesson has been found by several major studies to be one of the most effective means for student' learning (Hiebert \& Grouws, 2007). Recent years have been marked by increasing efforts to emphasize the conceptual aspects of mathematics together with an emphasis on students' agency and authority (National Council of Teachers of Mathematics, 2000). Professional development efforts have focused on helping teachers afford students opportunities to engage with cognitively demanding tasks, while clarifying important mathematical concepts and ideas (e.g. Boston \& Smith, 2009). Yet this effort has been constrained by lack of sufficient tools for examining the extent to which instruction indeed affords explicit attention to concepts. In this paper, we suggest examining the conceptual aspects of mathematics instruction through the Realization Tree Assessment (RTA) tool (Weingarden, Heyd-Metzuyanim, \& Nachlieli, 2017). Using the RTA, we inquire into the catalysts of explorative instruction.

## THEORETICAL BACKGROUND

We define explorative mathematics instruction as instruction that supports explorative participation in mathematical learning. Explorative participation (Sfard \& Lavie, 2005) is participation for the sake of producing mathematical narratives to solve problems or to describe the world. Such participation is contrasted to ritual participation, which main goal is pleasing others and which is characterized by rigid rule following and endorsement of results as "correct" according to external authority. Explorative participation is linked more broadly to the view of mathematical learning as the process by which students gradually become able to communicate about
mathematical objects (Sfard, 2008). These discursive objects are produced by discourse (or communication), and are made up of different "realizations" (ibid, p. 165). For example, the signifier $1 / 2$, the process of dividing a pizza into two pieces, and the process of shading 3 circles out of 6 , are all samed into the object "one half". Children often learn each of these realizations separately and only later come to relate to them all to one object. This is the heart of a process Sfard calls "objectification". Objectification, or talking about mathematical signifiers as "standing for" mathematical objects that "exist" in the world, is a major and necessary accomplishment for advancing in the mathematical discourse. A mathematical object can be visualized as a "realization tree" where complex objects are made of simpler ones. For example: a half is made of different realizations ( $1 / 2,0.5,50 \%, 3 / 6$ etc.) but the whole numbers making up these realizations also have endless realizations (3 apples, 3 fingers, etc.).
Recent years have seen increasing efforts to train teachers to teach towards explorative instructional practices, but the change in teachers' practices has been found to be a complex process (Heyd-Metzuyanim, Smith, Bill, \& Resnick, 2016; Spillane \& Zeuli, 1999). In particular, constructing tools for the detection of change in teachers' practices that would fit the ideas of a professional development for explorative instruction, is not a simple matter.

The Realization Tree Assessment (RTA) tool (Weingarden et al., 2017) was built in order to examine explorative instruction by assessing the extent to which students are exposed to different realizations of the mathematical object during the lesson. In our former work, we have used it mostly to visualize qualitatively differences between lessons based on an identical task. The usefulness of the tool to compare and rank the level of explorative instruction has not yet been explored. Such ranking can enable the examination of the relation between explorative instruction and other variables such as grade level or track.

In the present study we enhanced the RTA tool to provide a numerical view of explorative instruction. With this tool, we asked: how are realizations that are exposed in the classroom connected to opportunities to form narratives about mathematical objects? And how are these opportunities connected to grade level and track?

## METHOD

The study reported here was performed in the context of the TEAMS (Teaching Exploratively for All Mathematics Students) project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms, using the "Five Practices for Orchestrating Productive Mathematics Discussions" (Smith \& Stein, 2011) and "Accountable Talk®" (Resnick, Michales, \& O'connor, 2010). As part of the PD, the teachers were asked to implement a task they encountered and experienced as learners in the PD session. This task is called 'the Hexagon Task' and it asks students to describe the perimeter of a general "train" in a pattern of hexagon "trains" (See Figure 1):

Figure 1: The Hexagons Pattern
The task was chosen since it had previously been shown to be cognitively demanding for students, as well as productive for teachers' initial attempts to implement discussion-based instruction (Heyd-Metzuyanim et al., 2016). The Hexagon task's richness lies in its affordance to connect different algebraic expressions to a single visual mediator (the Hexagons), as there are various different algebraic expressions that express the desired perimeter. Therefore, the task also provides opportunities for "saming" the different realizations of the perimeter and opportunities for students' engagement with the mathematical concept of identical algebraic expressions.
27 teachers participated in the PD, and 23 of them implemented the Hexagon task. However, 10 lessons were excluded from the current study based on their language (Arabic) and another 3 lessons were not included due to technical reasons. Thus the analysis was performed on 10 lessons. Analysis of the RTA is preformed based on watching only the whole-classroom discussion part of the lesson. Usually, several views are required for completing a tree. However, we were able to code a video in around a ratio of $1: 3$ time of coding per time of video. This is much less work than preforming the analysis based on transcripts.
The RTA depicts the different realizations of a mathematical object as nodes in a "tree" (see Figures 2 and 3). We code the tree according to two criteria: (1) Coloring the realizations that were exposed to students during the lessons based on who articulated the realization (dark color $=$ student; light color $=$ teacher.) (2) Arches between the realizations are drawn where links between realizations were made during the discussion (continuous line $=$ link made by students; dashed line $=$ link made by the teacher). We quantify the data as follows (see Table 1): (1) Number of realizations: the total number of realizations that were colored. (2) Ratio of students' realizations: the number of dark realizations out of the total number of colored realizations. (3) Number of horizontal links: the total number of links that were made between algebraic expressions' and the visual mediators of the hexagons pattern. (4) Number of vertical links: the total number of links that were made between any other two realizations. (5) Ratio of students' horizontal links: the number of horizontal links that were made by students (continuous line) out of the total number of horizontal links. (6) Students' vertical links: the number of vertical links that were made by students (continuous line) out of the total number of vertical links. (7) Narratives about the 'saming' of the algebraic expressions branch: this criterion received a "1" if a narrative about the 'saming' of the mathematical branch of algebraic expressions appeared anywhere in the discussion and was offered by students and "0" if it was not. Such narratives were for example: "all those formulas are the same". We did not count under this criterion narratives of 'saming' not offered by students since practically all lessons included such a narrative authored by the teachers. In addition, for each lesson,
we specify the grade level of the class and the track in the form of: track / total tracked groups in the grade.

To be able to compare RTAs one to another, we ranked each lesson on the basis of the following formula: No. of realizations/maximum realizations in the sample + ratio of students' realizations + no. of horizontal links/max horizontal links + ratio of students' horizontal links + no. of vertical links/max horizontal links + ratio of students' vertical links + saming expression. All the above were divided by 7 (number of criteria) to arrive at a ratio of 0 to 1 . Thus 1 indicates the most "robust" RTA in the sample (max realizations, max links) and 0 indicates an "empty" tree (no realizations and no links).

## FINDINGS

We start by describing the RTA of two lessons. This will be done both to exemplify the method and to display contrasting implementations of the task. The first lesson took place in $8^{\text {th }}$ grade and was directed by Yarden. Yarden's class was the highest of 3 tracks in that grade (therefore, coded $1 / 3$ in track column, see Table 1, line 3). In Yarden's lesson, the students were exposed to 5 different realizations, 4 of which were explained by students (see Figure 2).


Figure 2: Yarden's RTA


Figure 3: Tamar's RTA

For example, one of Yarden's students, who presented the $2 * 5+4(x-2)$ realization, wrote this algebraic expression while relating to each one of the terms in the expression: "the 5 represents the 5 sides in each sequence (points to the 5 'external' sides in the rightmost hexagon). The 2 is to multiple it for the other side (points to the 5 'external' sides in the leftmost hexagon)". The (x-2) term and the multiplication by 4
were explained by pointing to the internal (connecting) sides of the hexagons. Although there was a substantial number of students' horizontal links (3/4), there were no vertical links at all. This means that although the students were exposed to different realizations and to the links to the visual mediator of each realization, there was no public 'saming' of those different realizations. The discussion thus had a "show and tell" feeling, where each student presented his or her solution but links between solutions were not made. Not surprisingly, 'saming' narratives were not found during Yarden's whole classroom discussion.

In contrast to Yarden's lesson, Tamar's students (see Figure 3, and line no. 9 in Table 1) were exposed to a greater number of realizations (7) and they explained most of the realizations $(6 / 7)$ themselves. Horizontal links between algebraic expressions' and the visual mediator of the hexagon pattern were made consistently and always by the students (6/6). In addition, three vertical links between realizations were made during the discussion. In particular, the students linked between two algebraic expressions: (1) $2 x+2 x+2$ and $4 x+2$, (2) $3 x * 2-2(x-1)$ and $6 x-2(x-1)$, and the teacher linked between the $4 x+2$ and the 'plus 4 ' realizations (explaining that each hexagon added to the train contributes 4 sides to the general perimeter).
Some of the vertical links were not declared explicitly but rather implicitly. For example: after one student explained the $4 x+2$ realization, another student presented the $2 x+2 x+2$ realization. The student started explaining this expression but another student stopped her and said, while laughing: "It's cheating, $2 \mathrm{x}+2 \mathrm{x}$ is like $4 \mathrm{x} . .$. I also have one [laughs] $4 x+1+1^{\prime \prime}$. Those implicit links mark the final part of the 'saming' process, where students have already 'samed' the realizations and have come to talk about them as being equivalent. In Tamar's lesson, where there were multiple vertical links, students also authored narratives about the saming of the general "branch" of algebraic expressions. For example, students concluded that "all expressions lead to $4 x$ $+2 "$. Such narratives were not observed in Yarden's lesson.

As a whole, the RTAs of Tamar's lesson thus show a deeper engagement with the concept of equivalent expressions as compared to Yarden's lesson. This, although at surface level, Tamar's lesson included quite a few realizations authored by students. Yet the main difference between the lessons could be seen in the number of links between realizations, and especially the vertical links, which signal the "saming" of different algebraic expressions. These differences led to the robustness of Tamar's lesson, which was quantified as 0.87 , compared to 0.42 in Yarden's lesson.

A similar analysis preformed on the other 8 lessons elicited several points of comparison as will be elaborated next (see Table 1).
Relation between realizations, links and 'saming' narratives: as a general trend, the greater the number of realizations presented during the whole classroom discussion, the more links (horizontal and vertical) can be seen in the tree. Though this may seem self-evident, this relation does not always exist. Some lessons (such as Yarden's) do include multiple realizations, yet links (horizontal or vertical) do not appear in them.

Thus, the presence of realizations in the classroom public sphere is not a sufficient condition for the opportunities to objectify. However, our small sample does hint that such presence is a necessary condition. Thus we see that in lessons where only few realizations were presented, hardly any links were made (though they could have been made even between few realizations) and even less narratives were formed about the "sameness" of the algebraic expressions branch. We conclude from this that the number of realizations that students are exposed to during the lesson is one essential catalyst for creating links and objectification.

| Lesson <br> no. | Grade | Track | Ratio of <br> students' <br> realizations | Ratio of <br> Students' <br> horizontal <br> links | Ratio of <br> Students' <br> vertical <br> links | Saming <br> expressi <br> ons | RTA <br> Robustn <br> ess |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 9 | $4 / 6$ | $1 / 4$ | $0 / 2$ | $0 / 2$ | 0 | 0.22 |
| 2 | 8 | $3 / 3$ | $0 / 6$ | $0 / 5$ | $0 / 3$ | 0 | 0.33 |
| $\mathbf{3}$ | $\mathbf{8}$ | $\mathbf{1 / 3}$ | $\mathbf{4 / 5}$ | $\mathbf{3 / 4}$ | $\mathbf{0 / 0}$ | $\mathbf{0}$ | $\mathbf{0 . 4 2}$ |
| 4 | 9 | $3 / 3$ | $4 / 7$ | $3 / 3$ | $0 / 5$ | 0 | 0.58 |
| 5 | 7 | $1 / 2$ | $2 / 7$ | $1 / 2$ | $2 / 5$ | 1 | 0.65 |
| 6 | 7 | $1 / 4$ | $4 / 5$ | $2 / 3$ | $3 / 3$ | 1 | 0.75 |
| 7 | 8 | No track | $4 / 4$ | $3 / 3$ | $4 / 4$ | 1 | 0.84 |
| 8 | 8 | No track | $5 / 5$ | $1 / 1$ | $5 / 5$ | 1 | 0.84 |
| $\mathbf{9}$ | $\mathbf{9}$ | $\mathbf{1 / 4}$ | $\mathbf{6 / 7}$ | $\mathbf{6 / 6}$ | $\mathbf{2 / 3}$ | $\mathbf{1}$ | $\mathbf{0 . 8 7}$ |
| 10 | 9 | $1 / 2$ | $7 / 7$ | $6 / 6$ | $2 / 2$ | 1 | 0.91 |

Table 1: results of the RTA's coding of the 10 lessons
Relation of grade level and robustness of the RTA: One would expect an increase in the robustness of the tree as the grade of the classroom advances. This, since students in the $9^{\text {th }}$ grade are expected to be more familiar with mathematical ideas related to the equivalence of algebraic expressions. However, as Table 1 shows, the connection between grade level and robustness of the RTA was weak, if existing at all. Thus, there were lessons in $9^{\text {th }}$ grade which were low in robustness (e.g. lines $1 \& 4$ ) and there were $7^{\text {th }}$ grade lessons which were relatively high in it (e.g. lines 5,6 ).
Track: Unlike grade level, the track of the classroom seems to have a closer connection with the robustness of the RTA. Low tracks (e.g. track 3 out of 3 ) figure prominently at the bottom part of the table (ranks $1,2 \& 4$ ) while the upper part contains only high tracks (track no. 1 out of 2 or 4 ) or classrooms that were not tracked. We interpret this finding as indicating that students sitting in low-achieving tracks had less opportunities for objectification than their peers in high-achieving tracks. Of
course, low-track students also authored less realizations, forming a vicious cycle that may perpetuate ritual participation in these tracks.
Types of lessons: We interpret the table as consisting of three types of lessons. The first are those that have sparse RTAs, and hardly any links. These lessons (such as 1 and 2 in our table) are characterized by low attention to concepts and low student authority as can be seen in the small number of realizations present, and the fact that any links, if made, are authored by the teacher. The second type of lessons are the middle-scoring RTAs. These (lessons ranked 3-6 in out table) often have multiple realizations presented and even multiple links. However, the relatively low ratio of links made by students shows that the teacher was "pulling" the classroom towards new realizations and new links. This may show that the classroom is learning something new and that the teacher is trying to insert new ideas. The final type are lessons that are characterized by high attention to concepts and high students' authority (ranking 7-10 in our table). These lessons have very consistent and high ratios of realizations and links, and students author almost all of them. These lessons may be very productive and show high levels of exploration. RTA robustness may also indicate that the students have become quite familiar with the mathematical object and that 'saming' had already previously occurred in that classroom.

## CONCLUSION AND DISCUSSION

Our main goal in this paper was to examine the opportunities for students' explorative participation during lessons. These opportunities include exposure to different realizations, encouraging students to create links ("saming") between realizations, and improving students' mathematical learning through the creation of narratives about the mathematical objects. The analysis of the 10 hexagon lessons using RTA afforded us the opportunity to better understand what catalyses explorative instruction: the exposure of students to broad numbers of realizations and to links between realizations. This exposure seems to be most productive when narratives and links are made by the students, not solely by the teacher.

Our findings indicate that students' grade and their level of familiarity with algebraic content has no relation to their explorative participation. Robust RTAs from $7^{\text {th }}$ grades show that even when students are not yet very familiar with algebraic expressions, they can offer multiple realizations and form links between them. The situation is less encouraging in low-level tracks, where we see much less student authority, less realizations and less links between them. Our worry is that students in such tracks receive less exposure to different mathematical objects, even when explorative tasks are offered to them. This findings continues previous studies showing the negative effects of tracking on students' explorative participation (Boaler \& Staples, 2008). Particular illuminating, in this respect, are the two heterogeneous classrooms in our sample, figuring high in RTA robustness. These show that it is possible, and perhaps even more fruitful, to implement explorative tasks in such classrooms.

The use of the RTA tool in this study continues our previous research (Weingarden et al., 2017) where we used it to qualitatively examine and visually represent different levels of explorative instruction. Here, we have shown its utility to compare numerically between a relatively big numbers of identical lessons. Of course, the possibility to examine lessons based on an identical task is quite rare. We intend to pursue the usefulness of the RTA to compare between lessons that are based on different tasks in future studies.

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