Encouraging Explorative Participation in Linear Algebra through Discourse-Rich Instruction - the Importance of Object-level and Meta-level Learning

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Table of Contents

List of Figures

List of Tables

Abstract

Learner-centered teaching supports student engagement in meaningful mathematics, student collaboration for sensemaking and equitable instructional practices. Studies have described implementations of learner-centered teaching methods in tertiary settings. However, these were more focused on students outcomes, rather than on the processes of learning. Moreover, some researchers claimed there are drawbacks to inquiry-based and discussion-based teaching. These include learning without an expert, distracting social interactions and ineffectual communication in groups. Thus, this study had two goals. The first was to adapt instructional practices, shown to promote discourse-rich explorative participation, to a university linear algebra setting. The second goal was to examine learner-centered tertiary teaching and the processes involved to better understand what supports and what hinders student learning in this setting.

I pursued these goals through the use of the sociocultural commognitive framework, which has operational tools for describing and analyzing mathematical learning processes. The framework's holistic treatment of content, as well as social and teaching-learning interactions, allowed the examination of whole-class discussions, and learner-learner interactions in workshops. The workshops were designed as extra-curricular enrichment for science and engineering students. Data analysis included, first, examination of the potential of tasks designed for these workshops to support explorative participation. Next, the extent that opportunities for explorative participation were taken up in the whole classroom discussion was studied. Finally, the learning processes in small groups of students were examined.

Based on a previously developed tool named the Realization Tree Assessment, I developed a tool called the Discourse Mapping Tree (DMT), for mapping the potential of the tasks through analyzing the subdiscourses that this task would invite. An extension of this tool, the Discussion Discourse Mapping Tree (DDMT), was used to map the actual implementation of the task in the whole classroom discussions. The learning processes in small groups were examined through an analysis of the mathematizing using the DDMT tool and commognitive tools. The communication channels in students' peer-learning episodes were also examined.

The DMT and DDMT offered the opportunity to distinguish between object-level learning, where students develop new narratives about familiar objects, and meta-level learning, where students make new connections between realizations of objects treated as different. The analysis revealed that the designed tasks afforded opportunities for both object-level learning and meta-level learning and, in most cases, the whole-class discussions included numerous opportunities for meta-level learning. However, the small group discussions did not support meta-level learning. Object-level learning in peer discussions was supported only when the students' communication patterns supported learning and the student's objectification process was sufficiently advanced.

This study has practical, methodological and empirical implications. Practically, the tasks can be used by other instructors and utilizing the insights from this project, new tasks can be developed. Methodologically, the operational method of examining tasks and mathematical discussions can be used for other topics and other levels. Finally, this study showed that lesson-design in inquiry-based teaching should be attuned to the difference between object-

level learning and meta-level learning, and the teaching methods chosen should be suited to the type of learning required.

List of symbols and abbreviations

RTA – Realization Tree Assessment

A tool used for mapping realizations mentioned during a discussion (Weingarden et al., 2019). An RTA is a visual representation of realizations of a mathematical object and the connections between them.

DMT – Discourse Mapping Tree

A tool developed in this study to map a mathematical object by the subdiscourses available. (Introduced in Section 4.3.1)

DDMT – Discussion Discourse Mapping Tree

A tool developed in this study, based on the DMT, to map the narratives mentioned during a discussion by the subdiscourses used. (Introduced in Section 4.3.2)

1 Introduction

The Talmudic sages of the Great Assembly codified the Jewish daily prayers and included the request to give our hearts the understanding to learn, to teach, to keep and to do (Melamed, 2003, Chapter V). This coupling of verbs was meant to emphasize that active involvement is necessary for productive teaching and meaningful learning (Sherlo, 2020). Additionally, the Bible exhorts us to teach each student according to his way (Proverbs 22:6), that is to choose teaching methods that consider the student's needs. This type of teaching is labelled active, learner-centered teaching in modern literature, and has been extolled and encouraged by researchers and practitioners in all types of educational contexts.

Learner-centered instruction, where students are engaged and involved in meaningful learning activities, includes active learning methods which seek to involve students in reading, writing, discussing, solving, analyzing, synthesizing, and evaluating problems. These have proven productive in all levels of mathematics education - K-12 (e.g. Schoenfeld, 2014) and tertiary (e.g. Chappell & Killpatrick, 2003; Cline et al., 2013; Talbert, 2014; Wawro et al., 2012). Studies encouraging such instruction in the tertiary level have posited the importance of student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking and equitable instructional practices (Laursen & Rasmussen, 2019). Along with the broadening of practical experience of learner-centered instruction in tertiary mathematics classrooms, the research community has started discussing the productiveness and suitability of incorporating these types of non-traditional teaching methods into lectures and tutorials in university mathematics (Shalit, 2021; Wieman, 2007). This discussion includes an international consortiums of mathematics departments which are developing a community for supporting change in university mathematics education (Gomez Chacon et al., 2021). The developed and implemented active, learner-centered workshops presented in this study are part of this effort.

In the present study, workshops were developed to encourage student explorative participation in linear algebra courses. Student engagement is crucial for all university mathematics courses, and in particular in linear algebra courses. A student's success in their first semester mathematics courses impacts on their confidence and self-motivation - important factors in student engagement and future success in university courses (Varsavsky, 2010). Varsavsky found that, independent of the level of the student's previous mathematical background, the more successful a student was in a first course in university mathematics, the more advanced mathematics courses the student subsequently enrolled in. On the other hand, failure in a first course could lead to complete student disengagement from mathematics. Linear algebra is a mandatory first year course in nearly all post-high school science and engineering programs, as it is an essential tool for all engineering and science students. Thus, student success in linear algebra courses is important both for the content, which furnishes students with tools they need for their future academic and industrial careers, and for the affective considerations towards their future academic engagement and success.

For the past 25 years I have been teaching tutorials in various university mathematical courses, mostly in linear algebra. Initially, my teaching practices included solving problems on the board to model problem solving techniques and formal proof writing. With experience, my teaching practices evolved to include more active learning and in-class discussions. Education courses intended for secondary school mathematics teachers exposed to me the benefits and advantages of active, student-centered teaching methods. Later, I became aware of the existence of a wealth of documented practices and research in this area. I thus searched for a means of incorporating these practices into existing linear algebra courses using a well-defined, methodical approach to support student explorative participation and student learning and to examine these practices.

In pursuing my goal to design and study these workshops, I needed a holistic theoretical framework that would allow me to analyze the processes involved in learning and the mathematical content concomitantly. The process, and not only the outcome, of learning is important as learning mathematics cannot be described merely by scores on an exam. Students can solve a task without comprehending the underlying mathematical notions, as evidenced by their inability to solve a similar task worded differently (Sweller & Cooper, 1985). Additionally, studying mathematical activity from a sociological perspective highlights the importance of social processes that influence student mathematical learning (Lave, 1988). The commognitive framework (Sfard, 2008) has a well-defined method of describing and analyzing learning processes in mathematical classrooms that studies learning in the context of the social interactions involved and the surrounding culture. This framework also allowed me to study processes of communication in the classroom, particularly from a holistic perspective attending to content and social interactions concomitantly (Heyd-Metzuyanim & Sfard, 2012). Thus, this is the theoretical framework selected to use in this study.

There were two main goals of this study. One goal was to adapt instructional practices, shown to promote discourse-rich explorative participation in elementary and secondary schools, to a university linear algebra course to support and encourage student participation and learning. Adapting these instructional practices included designing tasks and lesson plans aimed at promoting discourse-rich explorative participation in tertiary mathematics courses. These were implemented in discussion-based workshops in linear algebra courses in a science and engineering university. The second goal was to explore an implementation of the above adaptation to better understand the processes of learning in university settings in terms of the content and the social interactions involved.

In the following chapters I first present, in Chapter 2, the theoretical background including the existing research pertaining to learner-centered instruction in undergraduate mathematics education, to learning and teaching linear algebra and to teaching practices that support explorative participation. I next present the commognitive framework, the theoretical framework used in this study. In Chapter 3 I present the research goals and research questions, and the methodology used to answer the questions posed is presented in Chapter 4. The findings of this study are presented in Chapters 5, 6 and 7. Finally, in Chapter 8, I summarize the findings and the conclusions drawn from these findings pertaining to student explorative participation in linear algebra.

2 Theoretical Background

In this chapter I first describe learner-centered instruction in undergraduate mathematics and then focus on learning and teaching linear algebra. Next, teaching practices that encourage explorative participation are described. The commognitive framework is then introduced and explained. Finally, a short summary of the various aspects involved in this study – mathematical content, mathematical practice, student participation and their intertwined connection – is presented.

2.1 Learner-centered instruction in undergraduate mathematics education

Many undergraduate mathematics educators have been incorporating discussion-rich, active methods into their teaching, even before such methods were formally defined. Already in 1911 the disreputable Prof. R. Moore (Mahavier, 1999) documented his teaching style, which included student inquiry. More traditional, teacher-centered methods, such as lectures, are considered cost-efficient, as many students can be taught simultaneously, and exact formulations of concepts and processes can be presented to the students. Additionally, lecturing is a means of modelling mathematical discourses (Viirman, 2021). However, the traditional methods are not necessarily effective for robust learning. Biggs and Tang (2007) claim that the teaching methods in universities need to be learner-centered, because effective teaching necessitates that students be engaged and involved in learning activities such as relating, applying and theorizing – and not just memorizing (Biggs & Tang, 2007). Using learner-centered teaching methods requires a lot of time and effort from the lecturer, as well as the cooperation of the students (Legrand, 2001). Yet, the benefits of this type of teaching to the students' learning are manifold. Exam outcomes were positively impacted in first year engineering mathematics courses from deep learning behavior (Griese & Kallweit, 2017). Students learning in more learner-centered mathematical university courses displayed higher self-efficacy levels and experienced the learning environment more favorably (Lahdenperä et al., 2019).

Learner-centered teaching includes practices that support students actively taking charge of their own learning, becoming self-regulated, and developing their own study paths (Pepin et al., 2021). Learner-centered teaching practices can benefit student learning, attitudes, success and persistence in mathematics and related fields (Laursen & Rasmussen, 2019), among other positive outcomes (Griese & Kallweit, 2017; Ju & Kwon, 2007; Lahdenperä et al., 2019; Laursen et al., 2014). There have been various interventions in university courses attempting to provoke student engagement and involvement using learner-centered teaching methods. There are many studies focusing on active teaching methods in various scientific disciplines (e.g. Kimmel & Volet, 2010; Tal & Tsaushu, 2018). Some examples of various learner-centered methods incorporated in mathematics courses include classroom voting via technological aids (Cline et al., 2013), Project Based Learning (Talbert, 2014) and flipped classrooms (Love et al., 2014). These, and other active teaching methods, have been used to support student engagement, evoke student motivation and allow students to work on complex tasks with the active guidance of an instructor (Talbert, 2014). These were most successful, as measured by student academic success, when implemented in courses which also modified the curriculum and the assessment methods to accommodate the teaching method (Vithal et al., 1995).

The studies describing the implementations of learner-centered teaching in tertiary mathematics classrooms focus mainly on the teaching methods and the outcomes of these implementations. Below I describe in more detail some implementations that used group learning episodes and discussion as part of their learner-centered teaching methods in tertiary mathematics classes.

Rasmussen and Kwon (2007) implemented learner-centered instruction in an undergraduate differential equations course by designing inquiry-oriented instructional sequences with resources and teacher materials. They report that students' participation in an implementation of this contributed to a positive transformation in the students' beliefs about mathematics and themselves as participants in mathematical discourse (Ju & Kwon, 2007). Ju and Kwon posit that this was due to real context problems, the emphasis on students' own resources, the instructor's guidance in developing a sense of authorship and ownership of knowledge and the decentralized structure of the course. Moreover, they explain that these practices supported new understandings of students, at a higher level.

Discussion based teaching has been suggested as a learner-centered method that supports meaningful student participation in mathematical discussions. Tabach and colleagues (Hershkowitz et al., 2014; Tabach et al., 2020) used such an approach in both middle school and tertiary mathematics classrooms and analyzed them using a networking of two methods to examine collaborative mathematical activity in such a context. Both episodes included small group discussions and whole class discussions. The tertiary episode in a differential equations course was part of a larger study examining undergraduate student learning (Stephan & Rasmussen, 2002). This approach was used to support students' meaningful mathematical activity of recreating mathematical ideas in a bottom-up manner. Stephan and Rasmussen characterized the collective learning in this context in terms of the mathematical practices, specific to differential equations, that emerged during the many sessions in a complex, non-sequential, non-linear manner. Hershkowitz and colleagues (2022) also used this classroom procedure of discussion in small groups and then reporting to the class about properties of fractals. The discussion about these infinitely constructed objects demonstrated that this method was successful and sparked meaningful discussions that supported student learning.

In addition to the success of the various implementations described above, Laursen and colleagues (2014) compared more traditional teaching with Inquiry Based Learning (IBL), which is a learner-centered teaching practice that uses carefully designed sequences to invite students to work out meaningful, unstructured problems. They used quantitative analysis and found that there were greater learning gains in IBL and equitable results with respect to gender. Additionally, student interest and confidence levels rose in IBL courses, and this too also displayed no difference between genders, as opposed to non-IBL courses that usually display gender-based differences. They suggest that this is due to the different characteristics of teaching practices involved that support deep engagement with meaningful mathematics and collaborative processing of mathematical ideas.

Along with the advantages of learner-centered teaching, there are also some deterrents in active, group-based learning described in the literature. First of all, time constraints and financial resources must be taken into consideration when planning to incorporate such methods into university courses (Talbert, 2014). In addition, there is an inherent risk that learning without an expert might be arbitrary and not advance toward the planned goal (Vithal et al., 1995). Another factor inhibiting learning in group work is affective considerations. Unequal identities of gender, race and the like affect learning outcomes and collaboration in STEM education at university levels (Carlone & Johnson, 2007; Johnson et al., 2020).

To conclude, many implementations of learner-centered teaching, and specifically, discussion-based teaching and group learning activities, in tertiary mathematics have supported student learning, student interest, student confidence, student outcomes and student engagement. The studies documenting these examined aspects of teacher activity, adapted innovative instruction to the undergraduate level, and studied student thinking. They discussed positive academic outcomes and also positive outcomes such as meaningful participation, creation of new understandings, and meaningful mathematical activity. These studies rarely examine holisiticly the design of the tasks set by the instructor, the instructor's implementation of the design, the students participation, the mathematical content and the intertwining of all of these. Little is known about how these various aspects encourage explorative participation. This is particularly important in linear algebra, where less is known about how such participation is involved in tertiary level mathematics learning processes.

2.2 Learning and teaching linear algebra

Linear algebra is a compulsory first year course in nearly all post-high school science and engineering programs, as it is an essential tool for all engineering and science students. Students' success with their first mathematics courses impacts on their confidence and self-motivation - important factors in student engagement and future success in university mathematics courses (Varsavsky, 2010). Varsavsky found that, independent of the level of the student's previous mathematical background, the more successful a student was in a first course in university mathematics, the more advanced mathematics courses the student subsequently chose to learn. On the other hand, failure in a first course led to complete student disengagement from mathematics. Thus, supporting student learning in linear algebra courses, which are often the first mathematics courses students enroll in, is vital for the students' future academic career, both for the mathematical tools and content learned in the course and for the affective impact on their learning in other courses.

Some research has been done on learning and teaching university level linear algebra, including studies of the difficulties students face and the types of thinking necessary for a students to gain conceptual understanding (Britton & Henderson, 2009; Harel, 2002). Dorier and Sierpinska (Dorier & Sierpinska, 2001; Selinski & Rasmussen, 2014; Sierpinska, 2000) claim that lecturers should explicitly state that assorted mathematical representations are the same object so that the student is aware of the equivalence of the various representations being studied. If a student comprehends that all the representations are indeed the same thing, student understanding and conceptualization of mathematical structures is promoted (Grenier-Boley, 2014; Harel, 2002). Thus, the student comprehends that the structures can be transformed, represented in different ways, and considered as being -- or not being -- isomorphic (i.e. mathematically equivalent) to other structures (Hillel, 2000). As I will later review, this necessity for more conceptual understanding can be met by learner-centered instruction. Thus, student participation in mathematical discussions in a thoughtful and profitable manner may promote the comprehension of these representations (Sfard, 2008).

Students face difficulties grasping the skills and concepts in linear algebra courses (Chang, 2011). Even students who excel in other courses often have a difficult time with linear algebra courses (Dubinsky & Leron, 1994). Harel (2017) explains these difficulties by noting that learning linear algebra includes learning about mathematical objects that have many representations and techniques for manipulating them. These objects—including matrices, linear equations, vector spaces and linear transformations—are all related and connected. One of the main conceptual challenges for students is the necessity of learning and understanding concepts, rather than computational algorithms (Britton & Henderson, 2009). Linear algebra is characterized by many abstract mathematical concepts that have no visual representations with intricate connections between them (Talbert, 2014). Students tend to think mathematics consists of a set of distinct topics that are compartmentalized (Ang, 2001), but in fact they are intricately connected. Another challenge for students was that while solving problems, students focused on the limited procedures from what they remembered, and this inhibited other approaches being attempted (Lithner, 2000).

2.3 Teaching practices that encourage explorative participation

There are assorted ways that a teacher can encourage and support explorative participation. Explorative participation, which will be more accurately defined later in this chapter, is characterized by autonomous student participation whose goal is to author mathematical narratives. This section describes teaching practices involved in discussion-based learner-centered teaching methods, including the learner-centered ones briefly described earlier. This includes the tasks given to the students, the teaching practices that support a meaningful discussion, and group-based learning. These are discussed in the next sections.

2.3.1 Mathematical tasks that support explorative participation

Teaching mathematics includes posing questions – written or spoken, for individual or group work, for classwork or homework, and for assessment or formative pedagogical goals. These questions, or tasks, may offer more or less opportunities for students to engage with mathematical concepts, ideas, and strategies (Sullivan et al., 2015). Many inquiry-based tasks are aimed at constructing cognitive conflicts, commognitive conflicts or boundary objects (Sfard, 2021) to elicit from the students the need for new mathematical objects, for new rules or for amending familiar ones. Once the students are motivated to adopt new narratives, new objects and new rules, they still need to actually adopt them and engage with them. Therefore, tasks geared towards this are also necessary.

Various considerations are mentioned in the literature for choosing tasks included in tertiary classrooms promoting learner-centered teaching. These considerations include using real world problems (Chang, 2011), ensuring conceptual inclusiveness (Stewart & Thomas, 2009) and providing the students with the opportunity for engagement in disciplinary practices (Zandieh et al., 2017). These do not provide operational characteristics of the tasks themselves.

One of the important considerations of tertiary task design is a high level of cognitive demand (Tekkumru-Kisa et al., 2020). A high level of cognitive demand in K-12 schools was defined by Stein and her colleagues as "tasks that involve … the use of formulas, algorithms, or procedures with connection to concepts, understanding, or meaning" (Stein et al., 1996, p. 467). Previous studies, in K-12 levels, have shown that tasks that support discussions, as part of learner-centered teaching practices, should be aimed at expanding students' mathematical experiences and invite students to deeper engagement (Koichu & Zazkis, 2021). These also do not provide operational characteristics of the task itself.

Other studies briefly describe why they chose the specific tasks used. For example, Weingarden et al (2019) used "the hexagon task" in middle schools because it had multiple solution paths, was challenging and engaging for the students, and it was previously shown that it produces rich whole-classroom discussions. These do not describe what characteristics of the task support learning and how the characteristics mentioned do support discussions. Cline and his colleagues (2013) described multiple choice questions they used to provoke discussions in a linear algebra classroom as "difficult questions" that required "interpreting calculations". These terms are not well defined, and the 6 specific tasks provoked a discussion since the students did not answer homogenously. Yet what were the characteristics of these tasks and was the discussion academically fruitful or just voluble? Hershkowitz and colleagues (2022) describe how the task about fractals they used successfully provoked a classroom discussion. They suggest that the task supported an inquiry-based discussion since the task had a low entry point, that is it was simple to understand, but a high ceiling, that is the discussion's mathematical level was quite high. None of these reviewed studies, however, provided, in a well-defined manner, how these tasks provide opportunities for explorative participation.

Smith and Stein (1998) list characteristics of tasks of high levels of cognitive demand, that engage students in a manner that increases students' ability to think and reason. They explain that such a task requires complex and non-algorithmic thinking, requires students to explore ideas, demands self-monitoring, requires students to access relevant knowledge and analyze the task, and requires considerable cognitive effort. These characteristics mainly describe the implementation of the task, as seen by the wording that they require certain actions on part of the students. The implementation of the task is an integral part of the cognitive demand in this list of characteristics. Stein and colleagues (1996) distinguish between tasks that demand student engagement at various levels, where the deepest level of engagement includes interpretation, flexibility, shepherding of resources, and construction of meaning. That is, there are tasks, independent of the implementation, that can support more explorative participation and there are tasks, independent of the implementation, that support less explorative participation. In this study I searched for characteristics of the tasks themselves, as a necessary but not sufficient condition for the tasks being implementable in a manner that supports explorative participation. I needed operational criteria to support my design of appropriate tasks to use in the workshops.

2.3.2 Teaching practices that support discussion

Beginning a discussion with an appropriate task is necessary for a meaningful discussion, yet the instructor should also promote conceptual understanding and successful problem solving throughout the lesson (Smith & Stein, 1998). These teaching practices should include actively supporting meaningful mathematical participation, supporting student struggle in building understanding, emphasizing connections between procedures and concepts, and soliciting student thinking (Schoenfeld, 2014). Stein and her colleagues identified five specific instructional practices for implementing mathematical discussions in elementary and middle school classrooms as part of a launch, explore and discuss (LED) lesson. These lessons start with a launch of the topic and the task, give the students opportunity to explore the task in small groups and then discuss the solution in a whole class discussion. This framework encourages construction of ideas, guides student thinking and teaches mathematical discourse (Smith & Stein, 2011; Stein et al., 2008). These practices can be done in advance and during class discussions and offer teachers more control over the content of the discussion and lessen the improvisation inherent in orchestrating discussions based on various student ideas.

The five practices listed by Stein and her colleagues (2008) are *anticipating* what students will do, *monitoring* their work in class, *selecting* students' strategies that are worth discussing in class, *sequencing* students’ presentations, and *connecting* the strategies and ideas. Anticipating what students might do, what questions they might ask, and what difficulties students might have enables teachers to prepare for class more effectively. This also enables teachers to answer students with a well-thought-out answer that was prepared in advance (Fernandez & Yoshida, 2004; Schoenfeld, 1998). Teachers can include the mathematical ideas and strategies that students are using by monitoring their work, noting the students’ work, and identifying the strategies being used (Lampert, 2001). Monitoring also includes asking assessing and advancing questions, to focus students on the specific problem and to help students progress towards a solution (Schoenfeld, 1998). Once students’ ideas are identified and noted, a teacher can select those that are worth discussing in class. By selecting what to discuss, a teacher can direct the discussion towards the intended mathematical content and not be surprised by a different topic being discussed (Lampert, 2001). Sequencing the presentations in a logical way that builds a mathematically coherent story line can help students follow the flow of the ideas being introduced, maximizing the potential to increase student learning (Schoenfeld, 1998). Connecting the strategies and ideas discussed in class to each other and to the context allows a student to work at different levels, to obtain a sense of the larger picture, and to deepen understanding of a topic (Brendefur & Frykholm, 2000; Lampert, 2001).

Hiebert and Grouws (2007) stress the importance of connecting strategies and ideas in mathematics classrooms. They define teaching as interactions among teachers and students around content directed toward facilitating student achievement of learning goals, such that the most valued learning goals are student struggle and conceptual understanding. They suggest that conceptual understanding grows as mental connections become richer and more widespread. They suggest public noting of connections among mathematical facts to foster this. Noting connections among mathematical facts can be achieved by discussing the mathematical meaning underlying the procedures, by noting how different solution strategies are similar or different, and by reminding students of the main point of the lesson and how that point fits into the big picture.

Meaningful mathematical discussions must be instigated, guided and supported through appropriate moderation and talk moves (Michaels et al., 2008). Student mathematical activity is based on what is considered appropriate mathematical and social behavior in their specific classroom (Boaler & Greeno, 2000). There are mathematical and social norms that constitute the expected behavior in the classroom. Socio-mathematical norms are social norms that are specific to the mathematical aspect of the students’ activity such as what counts in the classroom as mathematically different, mathematically sophisticated, mathematically efficient, mathematically elegant and mathematically acceptable (Cobb & Yackel, 1996). For example, if a student asks why a statement is correct and the teacher responds, “that’s the formula”, then the socio-mathematical norm that “that’s the formula” is an acceptable justification is constructed. If, instead, the teacher explains why this formula is true, then the socio-mathematical norm of what is an acceptable and sufficient justification will be constructed differently. These norms are constructed by the expectations of the teacher, student responses, and the interactions between the two (Cobb & Yackel, 1996). Teachers can use verbal and facial cues to guide the conversation and to emphasize logical connections and reasoning (Michaels et al., 2008), which would support explorative participation.

2.3.3 Collaborative group learning

Many learner-centered methods, including discussion based teaching, include collaborative learning sessions in which various learners work together toward a common goal. It has been suggested as a classroom practice that encourages active learning and deeper engagement with the academic content and is a crucial working skill in the 21st century workplace (Barron, 2000).

In a university setting, collaborative learning, although less researched then in primary and secondary education settings, has been shown to promote positive social and academic outcomes (Cabrera et al., 2002). The benefits of collaborative learning cited in the literature include encouraging discovery, fostering student engagement, promoting student agency, advancing communication and collaboration skills, and fostering appreciation for many solution paths to a correct answer (Barron, 2000).

Yet, along with this long list of potential advantages, some researchers have pointed to the problems that can exist in student collaboration. These include distracting social interactions between members (Barron, 2000) and ineffectual communication (Nilsson & Ryve, 2010; Sfard & Kieran, 2001). Studies point to the existence of a connection between effective collaborative learning and the affective, social aspects of mathematical learning. The coordination between group members working together toward a common goal necessitates mutuality in the interaction, a shared task, and joint attention at critical moments (Barron, 2000). In middle school mathematics, it has been posited that ineffectual communication between participants can hinder mathematical progress (Sfard & Kieran, 2001). Motivational issues can also influence participation in a collaborative episode (Wood & Kalinec, 2012). Thus, collaborative learning sessions, although they have advantages, must be used thoughtfully.

2.4 Commognition

2.4.1 Choice of commognition as theoretical framework

The commognitive framework (Sfard, 2008) is a socio-cultural discursive theory, tailored specifically for mathematics, which enables the examination of the mathematical content and the learning processes involved in mathematical learning. This framework is inspired by several socio-cultural theorists, the most notable of them being Lev Vygotsky. Vygotsky (1978) explains that social interaction, such as interacting with other learners and with experts, fosters intellectual development. Studying mathematical activity from a socio-cultural perspective highlights the importance of social processes that influence student mathematical learning (Lave, 1988).

The commognitive framework has a well-defined method of describing and analyzing learning processes in mathematics classrooms. It has been shown to be productive for studying processes of communication in the classroom, particularly from a holistic perspective attending to content, social interaction and affect concomitantly in primary and secondary schools (Heyd-Metzuyanim, 2015; Heyd-Metzuyanim & Sfard, 2012; Sfard & Kieran, 2001). The commognitive framework was also found to be an effective tool for studying various aspects of university level mathematics (Nardi et al., 2014). The commognitive framework has also been used to study teaching processes (Nachlieli & Elbaum-Cohen, 2021; Viirman, 2013). This framework is thus appropriate for studying the varied aspects of tertiary mathematical classrooms – teaching, learning, participation and tasks – and their interconnections in a well-defined manner.

2.4.2 The main tenets of the commognitive framework

The commognitive framework defines learning as changing one's discourse, as part of becoming a participant in a certain community (Sfard, 2008). Discourse is defined within this framework as "set apart by its objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93).

Mathematical discourses are hierarchical and recursive, where their objects (e.g. rationale numbers in $\mathbb{Q}$) are built upon previously established objects (e.g. whole numbers in $\mathbb{Z}$) (Sfard, 2008). Sfard maintains that historically, new mathematical discourses were created either by several familiar discourses coalescing into one discourse or by a meta-level discourse *subsuming* an older one. This historical process, according to Lavie and Sfard (2019) may be reconstructed in the development of students' individualized discourse. When learners progress from one discourse to a subsuming one, the subsuming discourse includes an isomorphic copy of the old ones, as well as new objects and narratives that can only be realized in the new discourse.

Adapting this theory to the domain of linear algebra, one can observe that there are multiple discourses in this domain that first have to be adopted, and then coalesced into one single subsuming discourse. I exemplify this process on the mathematical notion of systems of linear equations (SLE). Historically, there are several different domains that represent SLEs, as described by Andrews-Larson (2015). Originally, systems of constraints on everyday problems were described verbally. Next, linear systems and their solutions were described by Chinese mathematicians in 200 BC and by Gauss (early 19th century) without matrix notation. Significant advances in notation, including matrices and determinants, led to SLEs being described as mathematical objects, and not merely as a process to a solution. This allowed SLEs to be represented by their properties. The modern, formal, axiomatic definitions of relations and operations on vectors utilizes vector spaces and linear transformations to describe SLEs. In accordance with this historical development of the representations of SLEs, we can divide the various narratives that can be authored regarding SLEs into five main domains - the solution set (constraints), a list of equations, matrix notation, properties of SLEs and vector spaces and transformations.

Nachlieli and Tabach (2012) theorized the learning of functions in middle school as first becoming proficient in multiple subsumed discourses, such as algebraic symbolic expressions, graphs and tables, and then becoming proficient in the unified discourse of functions. Similarly, I theorize that the learning SLEs, as part of a linear algebra course, includes becoming proficient in each of the discourses listed above and producing narratives within them, and ideally, then eventually coalescing these separate discourses into one unified discourse of SLEs. Following this theorizing of learning SLEs, this theoretical framework of learning can be used to describe learning for all topics in linear algebra.

2.4.3 Objectification

According to commognition (Sfard, 2008), learning mathematics involves familiarizing oneself with discursive objects that only exist in socially constructed discourse. The words or symbols that are used in the discourse are termed s*ignifiers*. *Realizations of a signifier* are expressions that are all interchangeable, which are all treated in experts' mathematical discourse as denoting "the same" object (Sfard, 2008). For example, the object "two", which is the number you reach when counting two apples and is a product of counting, can also be signified by the numeral 2. This object can also be realized using additional signifiers such as by 4/2, 1+1, and $\sqrt{4}$. Another example is the signifier "vector space" which can be realized by an algebraic expression describing its general element (e.g. $\{(x,y,-x-y) \mid x,y \in F\}$), by a linear span of one of its many bases (e.g. Sp{(1,-1,0),(0,1,-1)}) or by the solution of a homogenous system of linear equations (e.g. x+y+z=0). Mathematical discourse consists of narratives about realizations of signifiers and manipulation of those signifiers.

Objectification happens when students come to communicate about mathematical symbols (e.g. $\sqrt[8]{1}$) as representing objects in the world (e.g. "the set of complex numbers which divide the unit circle into 8 equal parts"). The objectification process includes substituting descriptions of actions and processes with descriptions of the products of these processes as if they occurred without the participation of human beings (Sfard, 2008). Saying "there are four cookies" is a reified restatement of "when I recite the counting chant and point to each cookie, I end up at four". We state that a triangle is encircled instead of saying that we drew a circle around a triangle. Instead of discussing "the values that a person plugs into a polynomial to achieve 0", we can discuss the "roots of a polynomial". Objectification eliminates both the process that created the object and the author of the process, allowing these objects to be discussed as if they exist regardless of human action.

One of the main challenges in learning mathematics stems from the need to form narratives about mathematical objects, which one has not yet objectified. When a student has not objectified the objects involved in the discourse, participation in that discourse can be done at first only by imitation of more knowledgeable experts (Sfard, 2008). *Saming* the different realizations of an object (e.g. 1 + i and $\sqrt{2}$cis(π/4)) is an essential step towards such objectification. This occurs when students come to see two or more realizations of a mathematical signifier as exchangeable and equivalent (Sfard, 2008). This is based on the fact that endorsed narratives using one realization (the parabola cannot meet the x axis in more than two places) is endorsable when translated to a different realization ($ax^2 + bx + c$ has at most 2 roots).

Endorsable narratives that can be samed can be from within a single discourse or from two disparate discourses (Sfard, 2008). Similarly, the saming of realizations of an object can occur between realizations from within the same discourse or from within different discourses. Weingarden and colleagues (2019) describe saming between two algebraic expressions *4n+2* and *6n-2(n-1)* of the perimeter of *n* connected hexagons. In this case, the signifiers, keywords and metarules are the same. These narratives are from within a single discourse and links between them are denoted as vertical by Weingarden and Heyd-Metzuyanim (2019). They also describe saming between a table of values and the above algebraic expressions to describe the perimeter of the aforementioned train of hexagons. The table of values is from within a separate discourse, as it uses different signifiers, different keywords and different metarules. This saming is denoted as horizontal by Weingarden and Heyd-Metzuyanim.

Saming realizations of objects with vertical links, within a discourse, extends the discourse. Saming realizations with horizontal links, in between discourses, authors narrative from the subsuming discourse, which is a coalesced discourse of all the subsumed discourses. This coalesced discourse includes narratives that are endorsable in all the subdiscourses and new narratives (Sfard, 2008). For example, the discourse of functions subsumes the discourse of algebraic formulas, of curves and of physical processes and includes narratives that can be endorsed in all of these discourses and narratives that include pieces of narratives from multiple discourses (Sfard, 2008, p. 174). This can also be exemplified in linear algebra in the complex numbers discourse. The narrative z = 1+i is a narrative from within the discourse of algebraic notation of complex numbers, which is a subdiscourse of the discourse of complex numbers. The narrative $z = \sqrt{2}\text{cis}(\pi/4)$ is from within the discourse of polar representation of complex numbers, another subdiscourse of complex numbers. Authoring the narrative that these are the same, i.e. $1 + i = \sqrt{2}\text{cis}(\pi/4)$, is a narrative from within the coalesced discourse of complex numbers, which includes the subdiscourses of algebraic representation of complex numbers and polar representation of complex numbers. It cannot be authored in any of those subdiscourses, as it is constructed of pieces of narrative that can only be authored in different subdiscourses. Thus, saming realizations of an object consists of authoring narratives in a new, coalesced discourse (Lavie & Sfard, 2019).

2.4.4 Mathematical routines and narratives

Mathematical learning, according to the commognitive theory, is the process whereby learners develop and refine their participation in the mathematical discourse by authoring narratives in familiar discourses and then, based on that, authoring narratives in new discourses (Sfard, 2008). These narratives include descriptions of mathematical objects and their properties and descriptions of manipulations of these objects (Sfard & Lavie, 2005).

Routines can result in narratives about properties of objects, such as finding the equation of a linear function from a table of values results in the narrative "the equation is y=ax+b" (Sfard, 2008) or finding a solution to an equation and stating, "the solution of $x^2=9$ **is** ± 3". Lavie, Steiner and Sfard (2019) define mathematical routines as a task and procedure pair used by a student to achieve a certain goal. These authors differentiate between the *task situation*, which is the way that a task-poser (such as the teacher) defines the task and the *task*, which is the way the task performer (learner) interprets the task. To exemplify this, examine the question posed by a teacher in middle school whether two given triangles are "the same" (Ben-Dor & Heyd-Metzuyaninm, 2021). The task situation was to determine if the triangles are congruent, and a student interpreted the task to be if the triangles are the same size. The procedure used by the student was to estimate the length of the sides of the triangle. Thus, the student's routine was to determine if the triangles are the same size by measuring them. Whereas the teacher's intended routine was to use geometric theorems to prove congruence. The students' learning processes can be examined through their routines.

An important distinction made in commognition around routines and the rules governing them, is between object-level rules and meta-level rules (Ben-Zvi & Sfard, 2007). Object-level rules deal with mathematical objects and how to manipulate them, such as how using scalars to multiply vectors would cancel them out. Meta-level rules, or metarules, define the patterns in the activity of the discourse and are custom-sanctioned, rather than externally imposed (Sfard, 2008). These are the rules about rules that constrain how to establish object-level narratives. Usually, metarules are variable, tacit, perceived as normative, constraining and contingent (Sfard, 2008). They can be rules pertaining to what type of answer is expected. For example, for a question starting with "how many", it is perceived as normative to answer with a quantifying phrase. They can be rules pertaining to what is considered a sufficient justification. For example, in a university classroom it is tacit that x+x = 2x, which would not be the case in middle school. Metarules become object-level rules once the discourse is adopted (Sfard, 2008).

Recent commognitive works have differentiated between two types of meta-level rules. Nachlieli and Elbaum-Cohen (2019) name them executive metarules and object related metarules. Sfard defined metarules as "*patterns in the activity of the discursants trying to produce and substantiate object-level narratives"* (Sfard, 2008, p. 201). This definition aligns with metarules for what is considered an acceptable proof and was labelled by Nachlieli and Elbaum-Cohen as an executive metarule. These type of metarules change and evolve. For example, the rules of what is considered an acceptable proof evolve from visual arguments in elementary school to formal, deductive proofs in university (Ben-Dor & Heyd-Metzuyaninm, 2021). Metarules are also defined by Sfard as "*rules that define patterns in the activity of the discursants*" (Sfard, 2008, p. 299), not necessarily about how to substantiate narratives. This definition pertains to rules specific to the object being studied. For example, multiplying by a whole, positive number makes the product bigger, which is not the case when multiplying by a fraction (Nachlieli & Elbaum-Cohen, 2019). Another example of a rule specific to an object is commutativity of multiplication. It is an object related metarule for authoring narratives about scalars. In contrast, this metarule is non-canonical in matrix multiplication. These types of metarules were labelled by Nachlieli and Elbaum-Cohen as object related metarules.

2.4.5 Object-level learning and meta-level learning

In object level learning, students gradually produce (or endorse) an increasing number of narratives about familiar mathematical objects (Sfard, 2008). Most learning occurs through object-level learning, however when new mathematical objects and new rules of discourse are introduced, the learning is meta-level and requires a change in meta level rules. Barnett (2022), using the commognitive framework, differentiated between types of learning by describing endogenous growth and exogenous growth. Endorsing new object level narratives within a discourse, when there is no change in metarules, is endogenous development. Exogenous growth involves the adoption, or significant modification, of metarules.

Within exogeneous development there is a distinction between horizontal and vertical development (Barnett, 2022), which aligns with Nachlieli and Elbaum-Cohen's (2019) distinction between types of metarules. Exogeneous horizontal development occurs when a number of previously separate discourses subsume into a single new discourse. Barnett exemplifies this with describing how modern graph theory subsumed electrical circuit design, recreational puzzles and map colorings. Exogeneous vertical development combines an existing discourse with its meta-discourse, that is with new metarules about proving. Barnett (2022) exemplifies this by describing Dedekind's creation of new ways of proving (by using new mathematical objects of ideals instead of a number). Meta level learning happens both vertically and horizontally.

Learning linear algebra, like all learning, is becoming fluent in the discourse of linear algebra (Sfard, 2008). This includes becoming fluent in all the subdiscourses of this topic (such as matrices, systems of linear equations, vector spaces, etc.) and saming the realizations from the various subdiscourses. Learning linear algebra involves object-level learning by authoring narratives within a discourse. For example, producing narratives about different matrices by using routines tailored for matrix manipulation (e.g. reducing a matrix to Echelon form). Learning linear algebra also involves meta-level learning, both vertical and horizontal. The vertical metalevel learning includes adopting more general mathematical metarules of proof and justification, rules labelled executive metarules (Nachlieli & Elbaum-Cohen, 2019). The horizontal meta-level learning includes adopting new coalesced discourses and object related metarules.

Previous studies (not within the commognitive framework) have stressed the importance of making students of linear algebra aware of the equivalence of the various representations being studied (e.g. Selinski & Rasmussen, 2014). This shows that horizontal meta-level learning, i.e. adopting new, coalesced discourses, is probably ubiquitous in linear algebra classrooms. An example of such learning in linear algebra would be familiarizing oneself with the routines of manipulating vectors as elements of vector spaces and saming these with the routines in the discourse of n-tuples. The meta-level learning required includes adopting the discourse of vector spaces and making the meta-level shift to the new meta-level rules in the new, coalesced discourse. In commenting on tertiary mathematics in general, Thoma and Nardi (2018) point out that first year mathematics courses include many meta-level shifts, due to the numerous new mathematical objects introduced, the rules governing their manipulation, and the metarules of formal proof that are unfamiliar to graduates of secondary school. This can be applied to linear algebra as well, which includes many new objects, new procedures and formal proof construction (Malek & Movshovitz-Hadar, 2011). Learning in linear algebra involves adopting executive metarules in vertical exogeneous development, adopting object related metarules in horizontal exogenous development and adopting new object level narratives in endogenous development.

2.4.6 Ritual and Explorative Participation

As reviewed above, learning mathematics, according to commognition, includes meta-level shifts to new discourses. The shift involved in meta-level learning can often be done at first only *ritually*, that is, by imitation of more knowledgeable experts (e.g. Sfard, 2007a). This ritual entrance into a discourse stems from the fact that the learner of a new discourse (e.g. $\mathbb{C}$) is faced with a seemingly impossible task of communicating about discursive objects ("complex numbers") that do not yet exist in his discourse. Ritual participation is characterized by manipulation of mathematical symbols focused on the procedure rather than on the final narrative about the mathematical object (Sfard & Lavie, 2005). The counterpart of ritual participation is *explorative* participation, which is characterized by taking part autonomously and creatively in the discourse. The goal of ritual participation is usually to please others, while the goal of explorative participation is to produce mathematical narratives (Heyd-Metzuyanim & Graven, 2015). Another hallmark of explorative participation is objectification of mathematical objects in the discourse, that is mathematical objects exist independent of processes and new narratives pertaining to these objects are authored (Sfard & Lavie, 2005).

Student participation gradually progresses from ritual participation to explorative participation (Lavie et al., 2019; Sfard & Lavie, 2005). A student who participates ritually in the conversation can implement memorized procedures but is not able to construct new narratives about the object nor to flexibly choose alternative procedures for substantiating a narrative about the object (Sfard, 2008). The differentiation between ritual and explorative behavior is not discrete, rather student mathematical actions can be characterized on a continuous spectrum between the two (Lavie et al., 2019; Viirman & Nardi, 2019). Continuous ritual participation, that does not evolve into explorative participation, generally produces mathematical failure (Heyd-Metzuyanim, 2015).

A necessary, but not sufficient, condition for explorative mathematical participation is appropriate opportunities to learn (Kilpatrick, Swafford, & Findell, 2001; Nachlieli & Tabach, 2019). Opportunities to learn (OTLs) are circumstances that allow the students to engage in and spend time on academic tasks. The task that is presented needs to be suitable, as different tasks will create different opportunities to learn.

2.4.7 Explorative Instruction

Explorative instruction is instruction that affords students opportunities for explorative participation (Weingarden et al., 2019). Such teaching has been described in mathematics classrooms in elementary schools (Baor, 2020), middle schools (Nachlieli & Tabach, 2019; Weingarden et al., 2017) and secondary schools (Nachlieli & Elbaum-Cohen, 2021). Explorative teaching was described as, "Teachers' actions that provide students with tasks that could not be successfully solved by performing a ritual. Rather, a successful completion of the task can only be achieved by participating exploratively" (Nachlieli & Tabach, 2019, p. 257). In contrast, teaching that affords only opportunities for ritual participation is characterized by instructional routines that focus on procedures and afford little opportunities for students to author their own narratives (Weingarden et al., 2019).

Weingarden and colleagues (2019) examined classrooms discussions and assessed them for explorative participation of the students. They mapped the realizations mentioned during a discussion; which links between realizations were authored during the discussion; and who authored these by using an RTA (realization assessment tool). An RTA uses the notion of an object being a "signifier together with its realization tree" (Sfard, 2008). It is a visual representation of realizations of a mathematical object and the connections between them. Once the mathematical object in the discussion is determined, an RTA can be constructed. This includes listing the object's realizations, determining the possible relations between them, grouping together realizations of similar types and determining which realizations and links were authored in the discussion and by whom.

Weingarden and colleagues (2019) describe a mapping of a discussion in a 7th grade class asked to describe the perimeter of a train of n hexagons. After the students worked in small groups for 35 minutes, they presented their solutions to the class. One student explained his group's solution of 4x+2. The student used a visual realization and pointed to a picture on the board depicting the train of hexagons. The student also used a verbal realization and described the perimeter of the train. Finally, the student used an algebraic realization and authored the narrative 4n+2. This student also explained the connections between these realizations and thus authored links between them. The RTA drawn for the discussion is in Figure 2-1, below.

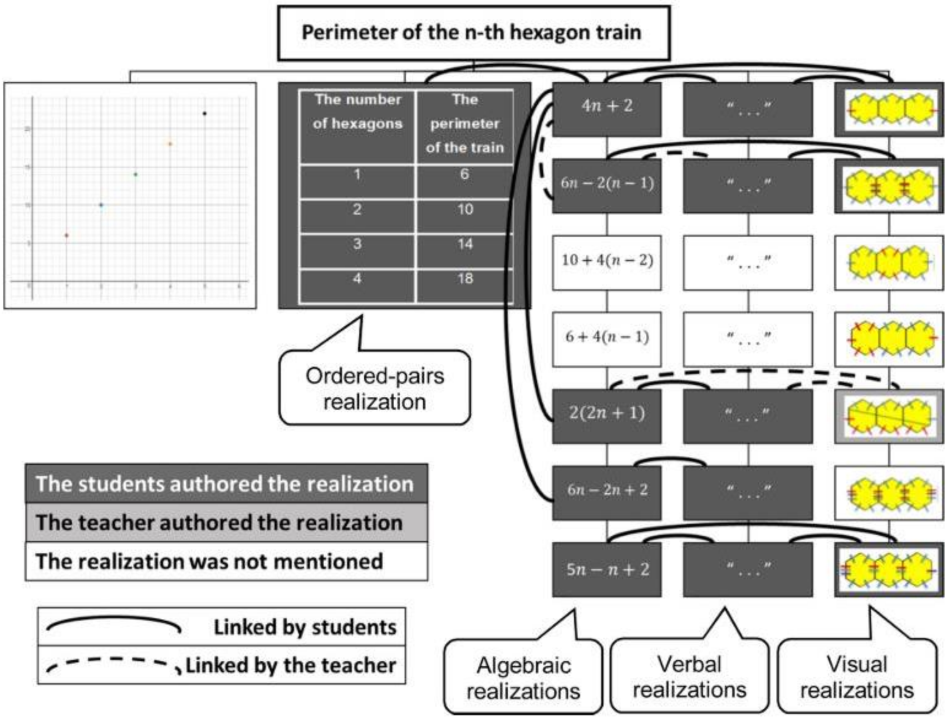

Figure 2-1 7th grade discussion mapped by RTA (Weingarden et al., 2019)

The RTA mapping affords an operational description of explorative instruction. That is, explorative instruction gives students opportunities and agency to construct narratives about mathematical objects, presenting different realizations for mathematical objects, and authoring links between these realizations. Similarly, the *potential* of a task for explorative participation can be operationally defined as the existence of the possibility of presenting different realizations for mathematical objects and the possibility for authoring links between these realizations.

Previously I described two types of saming realizations – within a discourse and in between two discourses. Authoring links in between realizations within the same discourse is object level learning, as this is adopting a new narrative within a discourse. Authoring links between realizations that are in different discourses is authoring a narrative from the new coalesced discourse, and is horizontal exogeneous development, that is meta-level learning. Learning mathemtics encompasses both object-level learning and meta-level learning, but explorative instruction has not been examined in detail with relation to object-level or meta-level learning. There are some suggestions that meta-level learning necessitates ritual participation, as a student cannot participate exploratively in a new discourse (Sfard, 2008). This needs more study.

2.4.8 Commognitive theorizing of group learning

As explained above, explorative instruction methods, like other learner-centered methods, often include sessions of small group learning, or peer learning. Several commognitive studies have tended to the issue of group learning, albeit often pointing to their weaknesses, rather than to their strengths. Sfard and Kieran (2001) showcased a pair of 7th grade students whose differing mathematical narratives did not lead to a meaningful mathematical conversation. Ben-Zvi and Sfard (2007) described another pair of 7th grade students whose mathematical learning did not advance in the group, despite one of the members of the group being an expert in the discourse the group was attempting to adopt. A similar story is recounted by Sfard and Chan (2020). These studies all pointed to the communication between the members of the group as ineffective and hindering the mathematical activity in the group.

Learning in a group setting includes the communication between group members about mathematical objects and the communication about the members of the group (Heyd-Metzuyanim & Sfard, 2012). In mathematics classes students are involved in mathematical discourse, these include attempting to describe objects' properties, finding solutions to equations, or achieving other mathematical goals. This is mathematical activity, or mathematizing. Yet there is always a concomitant activity going on, which relates to student identity, affective responses and how students position themselves in the discourse (Heyd-Metzuyanim & Sfard, 2012). These two activities are intertwined and ineffective communication can hinder the mathematical communication (Ben-Zvi & Sfard, 2007). Ben-Zvi and Sfard described a group learning session where the students' mathematical learning did not advance, despite one of the members of the group being an expert in the discourse the group was attempting to adopt, due to communication issues between the pair of students. Communication will be considered effective or ineffective based on if the responses of a pair of discursants are consonant with the pair's expectations (Sfard & Kieran, 2001).

The mathematical communication in group learning occurs in multiple channels of communication (Sfard & Kieran, 2001). The first channel is the intra-personal channel, which focuses on a person's own reasoning and ideas. This occurs when a person communicates with himself about his ideas, although it could be out loud. The second channel is the interpersonal channel, where the participants in the discussion are focused on their partner's reasoning and ideas. Asking for corroboration for a claim, giving corroboration for a claim, questioning another's claim and answering a question asked by another are all in the interpersonal channel. The interpersonal channel includes reactive utterances, where the utterance is a reaction to another's utterance, and proactive utterances, where the utterance is aimed at getting a reaction from the other participant in the discourse.

Learning is a change in a student's discourse and can occur through communication in the various channels of communication (Chan & Sfard, 2020). A learner's proficiency in any discourse can possibly advance whenever a student is exposed to narratives from within that discourse. This exposure can occur in the intra-personal channel of communication, where a student authors narratives from within a discourse to himself, or in the interpersonal channel of communication, when another student authors narratives from within the discourse.

2.5 Summary of theoretical background

Through the review of the literature, I showed that learner-centered, discussion-rich, active teaching methods supported student engagement and deep learning in all levels of mathematical education, and particularly in university level mathematics education.

Many implementations of learner-centered teaching, and specifically, discussion-based teaching and group learning activities, in tertiary mathematics have supported student learning, student interest, student confidence, student outcomes and student engagement. Explorative participation in mathematical discussions can be supported by appropriate tasks and teacher actions, including moderation that encourages such participation. The studies describing the implementations of learner-centered teaching in tertiary mathematics classroom focus on the teaching methods and the outcomes of these implementations, and less on the student learning processes involved.

3 Research goals and research questions

There were two main goals of this study. One goal was to adapt instructional practices, shown to promote discourse-rich explorative participation to a university linear algebra course to support and encourage student participation and learning. The second goal was to explore an implementation of the above adaptation to better understand the processes of learning in an undergraduate classroom in terms of the opportunities for learning that were picked up in both whole class and small group discussions.

Adapting explorative instructional practices included designing tasks and lesson plans aimed at promoting discourse-rich explorative participation in tertiary mathematics courses. These were implemented in discussion-based workshops in linear algebra courses in a science and engineering university.

The questions asked were:

1) What was the potential of the tasks designed for the workshops to support explorative participation and encourage student learning? That is:

> (a) What are the mathematical objects that can be exposed through the tasks, their different realizations, and the opportunities for explorative participation that can be afforded?
>
> (b) How do the tasks afford opportunities for adopting new meta-rules involved in the discourse of linear algebra?

2) To what extent were opportunities for explorative participation taken up in the whole classroom discussion and in what ways?

3) What were the learning processes in small groups of the participating students? Specifically:

> a) What were the students' initial mathematical routines authored to solve the proffered task? Did they change as a result of the interaction, and if so, how?
>
> b) What were the patterns of communication during the interactions? How did the patterns of communication afford or constrain the change in students' routine during the interaction?
>
> c) What objects and subdiscourses were involved in the interaction? How did these afford or constrain the change in students' routine during the interaction?

4 Methodology

This chapter describes the methodology used in this study. The research setting is described in Section 4.1. Section 4.2 describes the data source, including the framework and content of the workshops. The analysis of these workshops is depicted in Section 4.3. Section 4.4 comments about my dual role as a researcher and an instructor in the workshops being studied. An ethical statement is brought in Section 4.5 and the trustworthiness of the analysis is discussed in Section 4.6.

4.1 Research Setting

This study was conducted at a science and engineering university, where all the students have successfully passed advanced level high-school mathematics courses required for entrance. Students take a linear algebra course, a requirement for most science and engineering students, during their first semester, as it is a prerequisite for many other courses. The data is from three courses – Algebra 1m Winter 2019, Algebra 1E Spring 2019, and Algebra A Winter 2020 course. Algebra 1m and Algebra A are taught in Hebrew and Algebra 1E is taught in English as part of an International Engineering program. More details about the courses and the students are discussed in Section 4.1.2.

For this study, workshops were offered to the students in linear algebra courses. Linear algebra is traditionally taught in the university using frontal lecturing methods. The students have 4-5 weekly hours of lectures and 2-3 hours weekly of tutorial sessions. The lecturer defines the mathematical notions, shows characteristics, proves theorems, and gives examples. In the tutorial sessions, the teaching assistant (TA) shows worked examples of problems utilizing the theoretical knowledge discussed in the lectures. The workshops assumed that the students had participated in lectures and tutorials, and thus they were somewhat familiar with all the definitions and theorems presented in those. Homework assignments in the courses consisted of a computerized parameter-based homework system for technical problems and handwritten human graded proof questions. The workshops took into account that some of the students had worked on the homework problems and some of the students had not. The workshops were offered in addition to the regular lectures and tutorials and held in parallel to the lectures and tutorials.

4.1.1 Linear algebra workshops

The sessions were one academic hour and participation in the workshops was voluntary. In the Spring 2019 semester, where there were 30 students registered for the course, 5 extra points were awarded on the homework grade (which is 10% of the final course grade) to students participating in at least 80% of the workshops. This was done to encourage participation, as with such a small number of students registered in the course, there was a concern that not enough students would choose to come to the workshops. In the other two semesters, there were over 500 students registered in each course, so no extra encouragement was deemed necessary to ensure sufficient student attendance for the study.

The number of students participating in each workshop varied based on how many students were aware that a session was to be held that day, what topic the session was about, prior commitments of the students and other factors. Thus, the number of students in the workshops varied greatly, from 7 students to 60 students. These are displayed in Table 4.1 below. Some of the students were in all or most of the sessions and some students were only in a single session.

The lesson structure of the workshops was an adaptation of the launch, explore and discuss (LED) structure and Smith and Stein's (2011) suggested practices for orchestrating productive mathematics discussions described in the theoretical background. At the beginning of each workshop, there was a short (around 5 minutes long) introduction. This included a summary of definitions and theorems presented in the lectures and tutorials. These were written on the board and were available to the students throughout the workshop. Next, the students were given a worksheet with tasks to work on together in small groups of 2 or 3 students. I was the instructor and I walked around answering questions and asking advancing questions where it was needed. This part took between 15-20 minutes. Finally, the students presented their solutions to the class and a whole-class discussion was moderated by the instructor discussing the proffered solutions, connecting the various solutions suggested by the students, and discussing other related topics brought up by the students' questions and examples. This discussion was usually 15-20 minutes long. Detailed lesson plans were written for the workshops. These lesson plans included lesson goals, mathematical tasks, multiple possible solutions, possible student difficulties, advancing questions for each difficulty and questions for further discussion. The lesson plans can be found in Appendix A, Section 10.1.

As summarized in Table 4.1, 13 workshops were held over 3 semesters about 6 topics from the course syllabus. The first two workshops, during the Winter 2019 semester, were used to test the feasibility of this type of workshop and to support the initial design phase of the project. These two workshops were not recorded but they were described in a research journal. The initial workshops showed that students were willing to attend more classes, in addition to the official lectures and tutorial sessions, and raised the expectation that students in future workshops would participate in the type of discussions planned and would be interested in learning actively. In addition, these two workshops allowed the moderator to learn and practice the skills needed for this type of teaching and to receive feedback on the moderating from teacher educators who had experience with explorative instruction in secondary schools. The conclusions from these two workshops informed the planning and implementation of the recorded workshops.

Overall, 13 workshops were held between November 2018 and February 2020. Table 4.1 summarizes all the workshops, their topics, the language spoken in the workshop, the number of students in the workshop and the timing in which they were held (in terms of week out of a 13-week semester).

4.1.2 Participants in the workshops

There are different levels of linear algebra courses given for the different faculties. In the Winter 2019 semester the students were from the Algebra 1m course, which is considered the second level (out of three) of linear algebra taught at this institute. This course serves students of various engineering faculties, including electrical engineering, mechanical engineering, bio-medical engineering, and physics. The students participating in the workshops included a male and female, Jewish and Arab students, similar to the general student body in the institute. All the students in the workshops were first semester students.

In the Spring 2019 semester the students were from an International School of Engineering Mechanical Engineering program. The course was Algebra 1E, which is parallel to Algebra 1m. The students were from North America, South America, Europe and Asia. Most of the students' native language was not English. The students had taken introductory math courses in the first semester and were in their second semester of the program.

The students in the workshops in the Winter 2020 semester were from the Algebra A course. This course is for students learning towards a degree in mathematics, computer science and data science. It is considered the highest level linear algebra course at the Institute. The syllabus includes more proofs, more abstract objects (for example, finite fields) and more hours of tutorial a week than Algebra 1m. This course is geared towards first semester students, yet it includes numerous students repeating the course. Thus, there were likely some students in the workshops for which this was not their first semester.

Course	**Workshop No.**	**Label**	**Week of Semester (13 total)**	**Topic of Workshop**	**Number of Students**	**Language**
Algebra 1m Winter 2019	1	P1	4	Matrices	25	Hebrew
	2	P2	6	Systems of Linear Equations	10	Hebrew
Algebra 1E Spring 2019	3	S1	2	Complex Numbers	14	English
	4	S2	5	Systems of Linear Equations	9	English
	5	S3	8	Linear Dependence	12	English
	6	S4	11	Linear Transformations	12	English
	7	S5	15	Diagonalizable Matrices	15	English
Algebra A Winter 2020	8	W1	2	Complex numbers	60	Hebrew
	9	W2	4	Matrices	15	Hebrew
	10	W3	6	Systems of Linear Equations	10	Hebrew
	11	W4	8	Linear Dependence	24	Hebrew

	12	W5	11	Linear Transformations	7	Hebrew
	13	W6	13	Diagonalizable Matrices	25	Hebrew

Table 4-1 The Workshops, timing, number of students, topic and language

4.1.3 The tasks that formed the basis for the workshops

The basis for the workshops were the tasks with which the students engaged, and which formed the basis for the whole class discussion. As reviewed in the theoretical background, selecting appropriate tasks for supporting explorative participation in learning processes is a complex project with multiple facets. First, tasks should support student learning by being challenging so that the students engage with the tasks, but not frustrating so that the students do not disengage (Tekkumru-Kisa et al., 2020). That is, the level and content of the task must be suitable. Another critical feature of such tasks is that they should support explorative participation (Cooper & Lavie, 2021), that is they should afford students opportunities for saming and objectification of the mathematical objects embedded in them. In addition, solving tasks should involve opportunities both for object level-learning and for meta-level learning (Sfard, 2008). Based on these considerations, the tasks were designed and developed.

The tasks evolved considerably over the two years of the project for several reasons. Some of the tasks failed to instigate discussions. Additionally, the wording of some of the questions, intended to encourage discussion, confused some of the students. In some of the workshops, the initial questions sparked discussions about a different topic than intended. Thus, modifications and tweaking were necessary and were carefully documented and carried out using a modified design based research cycle (Prediger & Gravemeijer, 2019), as will now be described.

I employed a design research approach which utilizes cycles of design and practice. Each cycle includes holding a session, reflective analysis, and improving the design. The initial task design was based on personal experience, on input from colleagues and other expert teachers and from the relevant literature (Denzin & Lincoln, 2011). The reflective analysis examined two main facets of the tasks - student participation in the discussions and the mathematical content. The analysis led to modifications of the tasks. For example, the initial wording of the introduction to the task about linear dependence used in Workshop S3 was:

> Are the following statements True or False? If a statement is true, prove it. If a statement is false, give a numerical counter example.

One of the statements that needed to be proved was:

> If $\{u_1, u_2, u_3\} \subset V$ is linearly independent, $u_4 \in V$, then $\{u_1, u_2, u_3, u_4\}$ is linearly independent.

In the first implementation of this task the students gave simple counter examples, such as a linear dependent set including the zero vector. These examples did not support a meaningful discussion about linear dependent sets, and to spark such a discussion more "interesting" examples were introduced into the discussion by the instructor. Answering the above questions focused the need to compel the students to author these examples and the wording of the question was changed for Workshop W4 to:

> True or False? If a statement is true, prove it. If a statement is always false, give a numerical counter example. If a statement is sometimes true, give an example when it holds and when it doesn't hold.

In Workshop W4, using this modified wording, the students authored more varied examples including more types of sets that supported the discussion, and there was no need for the instructor to introduce more examples. The wording of some of the other tasks were also changed to clarify the task for the students.

The modifications also included adding more questions to some of the tasks. In one of the workshops, some of the students participating had already worked on similar homework problems and found solutions immediately, whereas some of the students had not yet participated in tutorials on the topic and found the tasks frustrating since they were not familiar with the procedures and theorems of the topic. This led to some of the tasks being modified to include more questions on different levels to accommodate the wide range of students who participated in the workshops.

The 7 modified tasks are displayed, analyzed and discussed in detail in the findings section in Chapter 5.

4.2 Data

This study explored the content, the social interactions, and the communication in the workshops. The data collected included the tasks and the recordings of the workshops.

4.2.1 Tasks

The design and development of the tasks were described in a previous section. The original tasks, the modified tasks and the considerations about the tasks were used as the data for the first research question of this study.

4.2.2 Recordings

The first two workshops of the Spring 2019 semester, Workshops S1 and S3, were held in the institute's Center for Promotion of Learning and Teaching recording studio. This center is equipped with high-quality cameras, audio recording capability and appropriate white board for clear images of the board. It was designed to produce recorded lectures and tutorials. The quality of the recordings is very high, however only the board was recorded, and the small group interactions were not. Moreover, seating in groups was almost impossible in that room and the whole room was designed purely for frontal lectures. Therefore, I chose to trade recording quality (of whole class discussions) with appropriate physical settings for explorative instruction and the 9 other workshops were held in classrooms that allowed seating in groups, or at least in pairs. In these classrooms there was a stage for the instructor with a large whiteboard, the desks were bolted in place and the chairs were connected to the desks. To record discussions, four cameras on tripods were used. One camera recorded the activity at the board and 3 additional cameras were randomly placed in the classroom to record small group interactions. The whole class discussion was recorded from 11 workshops and 20 small group interactions were recorded.

4.2.3 Choice of groups to analyze from the small group discussions

The learning processes of students in small groups were examined by using discourse analysis. This is a highly work-intensive method and therefore, a principled data reduction process was needed, to choose the small group discussions that would be most illuminating for answering the research questions. This data reduction process will be described below.

As a first step, all the recordings were viewed and briefly summarized as to the mathematical content and the group dynamics. The twenty group recordings were labelled based on the workshop (S = Spring, W = Winter; Sn or Wn, n = number of workshop; Sn-i, i = number of group). For example, W3-2 signifies the second group recorded in the third workshop of the winter semester. A table summarizing this is in Appendix B, Section 10.2.

Next, I chose on which groups to focus more deeply. I wanted to examine the mathematical content of the small group discussions. Therefore, the groups that did not include enough visible mathematical activity were not further examined. Some of the mathematical activity was not accessible due to various factors such as unclear speech or students speaking in a language I do not understand. In some of the groups the students did not interact out loud very much, and mostly each student solved the task individually. In some of the groups the students did not justify their claims, either stating, “It’s obvious” or just not explaining their ideas out loud. In all, in 7 groups there was minimal audible mathematical discussions, thus they were not further examined.

Even after removing the above data, there were many groups and interactions left to examine. In order to examine the processes involved in collaborative learning, I searched for groups with contrasting interactions, as contrasting cases allows for comparison and contrast of the cases and thus gives a deeper look at each case (Meyer, 2001). One group whose interaction included equal participation was chosen and one group with unequal participation was chosen. The table below displays how these groups were chosen.

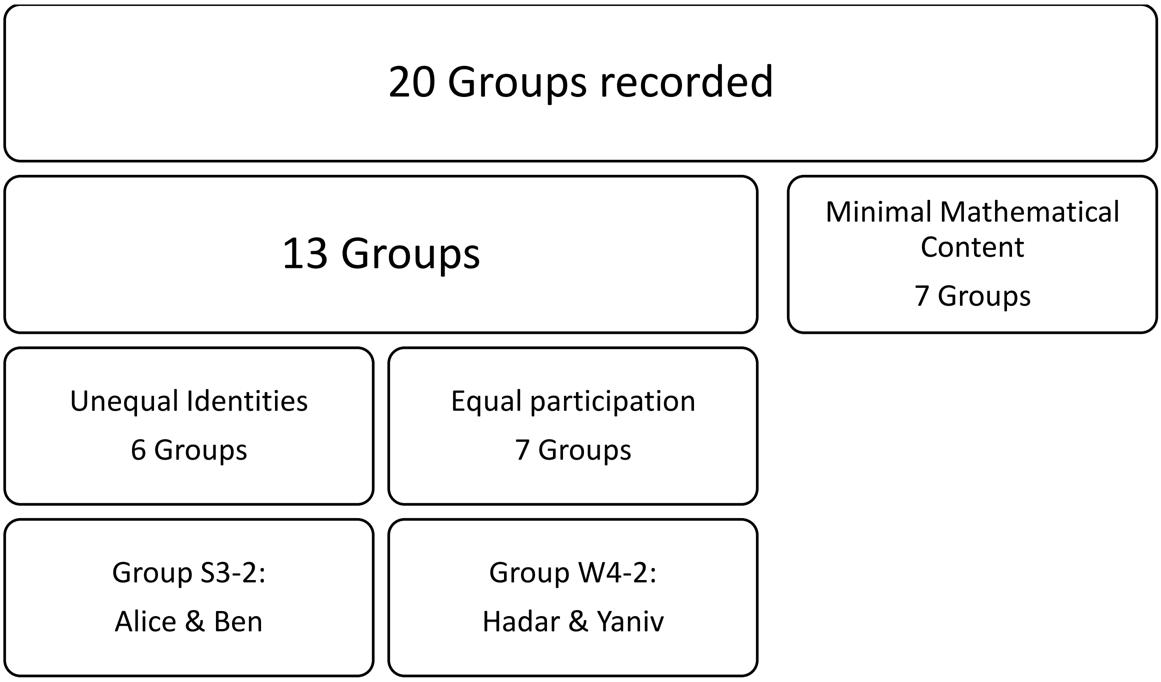

Table 4-2 - Group choice

4.2.3.1 *Choosing a group with a seemingly successful collaborative learning*

Group W4-2 was chosen as an illustrative example of a collaborative learning process. I searched for a group with equal participation to examine the processes involved in collaborative learning. There were 7 episodes that seemingly had interesting mathematical discussions to analyze, and the participation was mostly egalitarian, therefore the learning seemingly was collaborative. These 7 episodes were transcribed and examined more fully. The group W4-2 (groups labelled as described above - workshop W4, group 2) was a mixed gender pair given the pseudonyms Hadar and Yaniv. They explicitly disagreed at the beginning of their interaction and their discussion included initial non-canonical statements and seemingly a collaboratively constructed canonical narrative. They both authored narratives, they both questioned the other's narratives and they both seemingly advanced in some aspects of solving the task. This seemingly productive, joint interaction could shed light on the processes involved in a successful, collaborative learning episode. Thus, the dyadic interaction between Hadar and Yaniv was analyzed in depth to examine the processes involved in collaborative learning within the workshops.

4.2.3.2 *Choosing a group with a glaringly unequal communication*

Group S3-2 was chosen as an illustrative example of the learning processes involved in a pair with unequal communicational patterns. This is common in peer-learning (Barron, 2003), and there were 6 such groups, from among the 20 recorded. In these groups one of the pair acted as an expert and as a leader, and the other partner acted as a follower. The "expert" partners told the other student explicitly what to do, and the other member of the pair acquiesced to this and treated the first student as an expert. The expert member of the pair was considered the arbitrator of mathematical correctness. The follower either asked, "Right?" about a statement or waited for the expert's permission to continue. In some of these pairs, the expert also determined non-mathematical behavior. A follower asked, "Should we write it out?" and

"Do we need to give an example?" asking the expert how to continue. There are also groups where someone attempted to act like a leader, but the rest of the group did not follow his lead. In those groups a more equal discussion occurred, and those groups were not considered here. In the groups where the interaction included an "expert" and a "follower", the discussion was less equal. This allowed me to examine the processes of mathematical learning in an unequal interaction.

In Group S3-2, a mixed gender pair with an unequal interaction, difficulty agreeing on a proof was observed. This pair, Ben and Alice, was also noted in the teaching journal. The pair's final solution was non-canonical, and Alice presented it to the class while stating, "I don't agree with this". This incongruity between Alice's comments and her actions offered an opportunity to examine the processes involved when the mathematical activity is hindered by the social interaction.

4.3 Analysis

4.3.1 Commognitive analysis of the tasks and their potential for supporting explorative participation

Seven tasks were designed for the workshops to achieve the first research goal of adapting learner-centered practices to an undergraduate setting. I examined their potential for supporting explorative participation to answer the first research question (RQ 1) by first asking what are the objects that can be exposed through the tasks, their different realizations, and the opportunities for saming that can be afforded. To answer this question, I developed the Discourse Mapping Tree (DMT). This is an adaptation of Weingarden and colleague's (2019) Realization Tree Assessment (RTA) tool, which was based on Sfard's (2008) notion of realization trees and explained in the theoretical background. The RTA was used to map the engagement with mathematical objects in discussion-based lessons by mapping which realizations were mentioned during a discussion and which links were constructed. In contrast to the RTA, the DMTs were constructed as a way of examining the *potential* of a task, independent from the implementation of the lesson.

4.3.1.1 Constructing the DMTs

The DMT is based on the notion of an object being a "signifier together with its realization tree" (Sfard, 2008). It is a visual representation of realizations of a mathematical object and the connections between them. The first step in constructing a DMT is determining the root node that is appropriate for a certain task, which is not necessarily straightforward. This is since, first, the object at the center of the task is not always stated clearly in the task. Second, theoretically, all realizations of an object are equivalent and thus any realization can be the root node. For convenience, the root node was chosen as the title given to the central object of the task, as it is given in textbooks (for example: "Complex number" or "Diagonalizable matrix").

The mathematical objects determined to as the node of DMTs are families of objects, unlike the RTA and realization trees which use single objects as the node. For example, the RTA might use the object f(x) = 2x-4, whereas the DMT uses "linear functions". This modification was necessary as the tasks and discussions in the workshops involved families of mathematical objects, and not specific mathematical objects.

The next step of constructing a DMT is listing the object's realizations and grouping together realizations of similar type. I used the theory and definitions given in textbooks to find realizations and also examined multiple solutions to find more. Additionally, student discourse from the workshops, from tutorials, from midterms and exams, from homework sets and from questions posed over the course of the semester proffered many realizations. Each type of realization usually belongs to a certain discourse; thus, it has its own keywords, its own narratives and its own routines of manipulation. Each type of realizations was placed in a separate branch of the DMT. This process is detailed in the findings section. Below is an example DMT for the mathematical object "complex number", which can be realized in the subdiscourse of algebraic representation (e.g. *3+4i*), in the subdiscourse of geometric representation (i.e. a dot on a plane), in the subdiscourse of the polar, or trigonometric, representation (e.g. *5cis53.13°*), in the subdiscourse of $\mathbb{R}^2$ (e.g. (*3,4*)), or in the subdiscourse as the root of a polynomial (e.g. a root of $p(x) = x^2-6x+25$). Each one of these discourses is represented by a branch of the DMT shown below.

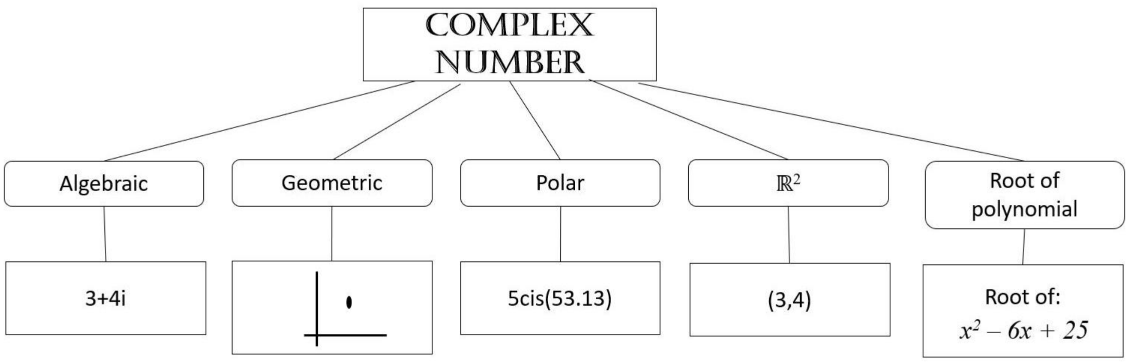

Figure 4-1 - DMT for "complex number"

Once DMTs are constructed for a designed task, they display the objects that can be exposed through the task, its different realizations, and the opportunities for saming offered by the task. The DMTs showed whether solving the tasks included multiple branches.

4.3.1.2 Micro-analysis of tasks to examine if the task necessitated the use of multiple realizations and multiple subdiscourses

The DMTs showed whether the use of multiple branches was possible to solve the tasks, yet they did not show if solving the task necessitated this. The extent to which a task demanded the use of more than one (object-level) branch was analyzed by a micro-analysis of the mathematics involved in the tasks. First, the possible solutions were discussed and approved by mathematical experts as possible that the task could be solved using this path, as probable that a student would suggest such a solution, and as correct mathematically. Each step of these solutions, or routines, was characterized by within which discourses they were authored, and which realizations of the object they utilized. This mapped the routine onto the constructed DMT and displayed if the suggested routine traversed multiple discourses. Next, the task was examined if it could be solved based on a single discourse or whether it necessitated traversing multiple discourses. This was done by studying the application of familiar routines, taught and rehearsed in the course, to the task at hand.

4.3.2 Examining extent opportunities for meta-level learning were taken up

I used recordings of the whole classroom discussions to answer the second research question (RQ 2). These were examined to study to what extent the potential of the tasks were taken up in the implementations in the workshops by mapping which subdiscourses were mentioned, which connections between subdiscourses were authored and who authored these. For this, I used the DMTs to construct DDMT (Discussion Discourse Mapping Tree), basing the procedure for this on what Weingarden and colleagues (2019) used for mapping middle school classrooms. My goal for mapping the whole class discussions in the workshops was to examine if there were realizations from within different discourses and if connections between these discourses were authored.

Mapping a discussion through construction of a DDMT included both a priori and a posteriori components. The branches of the DDMT for a workshop were drawn a priori using the branches from the DMT constructed for the object embedded in the task given to the students in that workshop. The branches available in the DDMT are the subdiscourses available within which object-level narratives can be authored about this object. This is exemplified for the discussion in Workshop W1 about complex numbers. The initial DDMT for this workshop was as below in Figure 4-2.

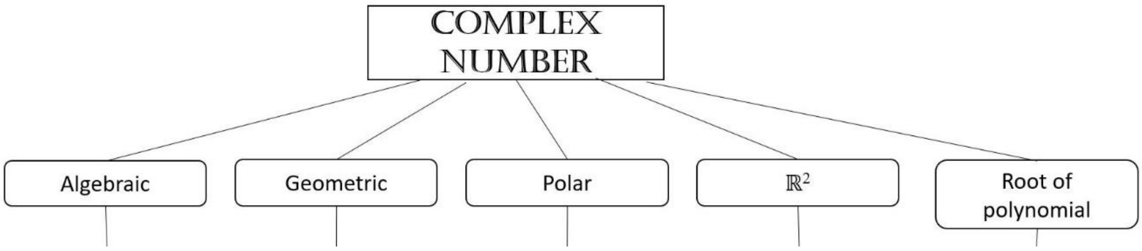

Figure 4-2 Available discourses for DDMT

The realizations shown on the DDMT were drawn a posteriori based on the realizations mentioned during the discussion in class. The drawing and classifying of realizations and links are exemplified on the DDMT constructed for Workshop W1 about complex numbers. In that DDMT a realization was drawn when a student wrote on the board that *$(2+3i)^2$ = $(2+3i)(2+3i) = 4+12i -9 = -5 +12 i$*. The narrative authored by the student is within the algebraic representation subdiscourse, as indicated the keywords, such as 2+3i, and metarules, such as $(a+b)^2 = a^2 +2ab +b^2$, from that discourse. The realization drawn is the mathematical object that the narrative describes and manipulates, which is $(2+3i)^2$ as a product of two terms. Thus, this realization, $(2+3i)^2 = (2+3i)(2+3i)$, was drawn on the DDMT on the branch of the algebraic subdiscourse and is labelled *I* in Figure 4-3, below. Following the student's explanation of what was written on the board, I (the instructor) asked, "*How can we calculate $(2+3i)^{17}$*?" This narrative is also within the algebraic subdiscourse yet has a different object – the complex number which is the outcome of $(2+3i)^{17}$. Therefore, this realization, labelled *II* in Figure 4-3 below was drawn.

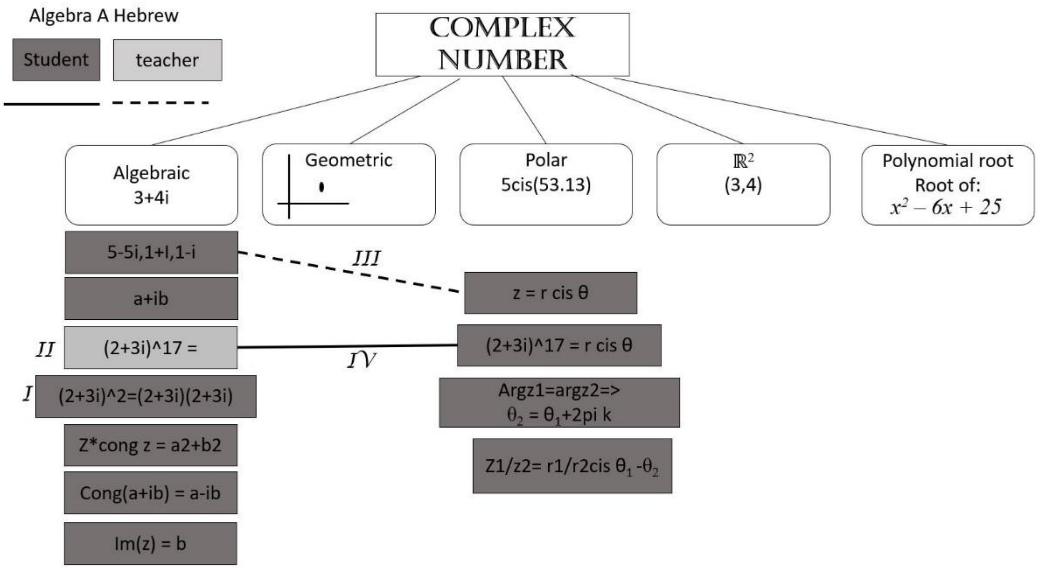

Figure 4-3 - DDMT W1 - Complex numbers

Adapting the methodology used by Weingarden and colleagues (2019), originally used on RTAs, the realizations were shaded in dark gray if a student authored the realization and light gray if the instructor authored the realization.

In addition to the realizations, I marked whether there was any saming between different types of realizations and who authored these. Horizontal links between the different branches of the tree were drawn. A solid line was drawn if a student authored a link between realizations, and a broken line was drawn if the link was authored by the instructor. For example, when I asked the class, "*How can this complex number (pointing to 1+i on the board) be represented in its geometric form*?" This offered the students the opportunity to author a realization of $1 + i$ in a different subdiscourse and to connect it to a realization in the algebraic representation subdiscourse. The student authored the realization $r\ cis\ \theta$ in the polar subdiscourse. The link authored between these realizations was then marked in the DDMT (line *III* in Figure 4-3, above). Answering the question, "*How can we calculate* $(2+3i)^{17}$?" a student said, "*We can change it to polar representation and then use de Moivre*('s formula)." This narrative connected between a realization in the algebraic and a realization in the polar subdiscourse (line *IV* in Figure 4-3).

The DDMTs provided an image of the whole class discussions that allowed me to examine the characteristics of the discussion and to what extent opportunities for meta-level learning were taken up. This is detailed in the findings section. The specific realizations that were mentioned were less crucial for this analysis. The specific realizations indicate object-level narratives, from within a subdiscourse. The object-level narrative used is an integral part of the meta-level learning but are not the focus of this analysis. The analysis focused on the subdiscourses used and the connections between them. The DDMT maps this, while also displaying which realizations were authored and by whom.

4.3.3 The intertwining of mathematical narratives and communication

The third research question (RQ 3) pertains to examining the learning processes in small group discussions without the support of an expert. These were analyzed for the mathematizing and for the communication patterns.

4.3.3.1 Analyzing mathematical discourse

The communication in a mathematical classroom includes mathematical narratives that relate to the objects, routines and mediators, and narratives that relate to other subjects or to other people (Heyd-Metzuyanim & Sfard, 2012). The learning process involved in the small group sessions were first examined through the students' mathematical routines.

The interaction was analyzed for its mathematical content by examining the mathematical routines as a task and procedure pair (Lavie et al., 2019). The mathematical routines used by the pair were delineated, and the task and procedure pairs were determined. The tasks each student was solving were established from the narratives they offered, and incomplete statements were filled in, using prior and subsequent statements. This was determined for the initial, individual routine of each student, when available, and also for pairs' co-constructed mathematical routines to examine if and how the interaction modified their mathematical narratives. For the co-constructed routines, who authored each mathematical statement and who adopted each statement was determined. This also allowed me to examine how the students' mathematical routines were modified. Once the pair's implementations of the problem-solving routine were established, they were compared to ascertain if they were mathematically aligned, that is, if they were consistent to an expert, external observer.

The mathematical narratives were also analyzed to differentiate between object level narratives and meta level narratives to examine if the learning process was impacted by whether the communication was around object-level or meta-level rules. These included implicit mathematical narratives that were not declared verbally by the participants but were implied by their verbal statements. Additionally, the objectification process of the students was examined to support the analysis of their mathematical activity. This analysis showed a detailed depiction of the mathematical activity involved in the pairs' interactions.

4.3.3.2 Analyzing the communication in a dyadic mathematical discussion

The learning processes in small groups is intertwined with the students' communication patterns, so these were next analyzed to allow me to examine the patterns of the students' communication. The communication was analyzed by studying the channels of communication used by the students for their mathematical communication. The students' discourse, as in all dyadic interactions, occurred in two channels simultaneously (Sfard & Kieran, 2001). The first channel was the personal channel, which focuses on the student's own reasoning and ideas. The second channel is the interpersonal channel, where the participants in the discussion are focused on their partner's reasoning and ideas.

To analyze the students' channels of communication, the transcript of their discussions about the tasks were first segmented into mathematical narratives. This allowed the examination of how each pair listened to each other's mathematical ideas - if they were attending to the mathematical content of each other's narratives and how they were actually participating in the discussion.

The narratives were classified as either occurring in the private channel, the interpersonal reactive channel or the interpersonal proactive channel. The utterances were classified as in the personal channel when one of the students communicated with him/herself about his/her ideas, although it could be out loud. For example, while a student was attempting to figure out a solution to a task, he stared at the paper or at the ceiling and stated his ideas out loud. He was communicating to himself about his ideas and the presence of the other student was ignored. This utterance was marked as having taken place in the personal channel. The interpersonal channel of communication included students asking for corroboration of a claim, giving corroboration of a claim, questioning another's claim and answering a question asked by another student. The interpersonal channel includes reactive utterances, where the utterance is a reaction to another's utterance, and proactive utterances, where the utterance is aimed at getting a reaction from the other participant in the discourse. These were all marked in the transcripts of the pairs' discussions. This classification allowed me to examine how the patterns of the students' communication supported or hindered change in their mathematical routines.

4.4 Trustworthiness

This study used qualitative methods of analysis and the commognitive framework within the relativist-constructivist paradigm, which maintains that knowledge and learning are a construct of human social interactions (Denzin & Lincoln, 2011). The trustworthiness of a qualitative study is gauged by its credibility, dependability, transferability, and confirmability (Denzin & Lincoln, 2011). In this project we used prolonged engagement with the data, expert debriefings, and a rich description of the data to establish the trustworthiness of the findings. The recordings of the discussions were studied extensively and repeatedly. The findings were discussed with experts in commognitive analysis and experts in mathematics. The data collected was presented in detail.

Some threats to the trustworthiness of the study derive from my studying the learning processes in workshops that I designed, planned and implemented as the instructor. A participant in the sessions acting as a researcher is a complex situation that might influence the subjectivity of the analysis and might compromise the expected role as a participant. Additionally, the students were aware that the workshops were being recorded, and thus their behavior might have been artificial. The cameras placed around the room could have inhibited students from talking freely. This too needs to be addressed.

My dual role as both a participant in the workshops and as an observer to the workshops is termed a participant observation (Rensaa, 2018). Rensaa suggested that participant observation reduced interference in the general running of the course and was more natural to the students than non-participant observation. Familiarity with the students and the classroom seemed to lead to the students behaving more naturally. Introducing an external observer into the classroom would have influenced both the students and me. Digital recording minimizes the intrusion from classroom observations (Wragg, 2011). Additionally, the observer role was minimized during the actual workshops, since the video recording freed me from the necessity of remembering what happened. I was too involved in teaching to take notes during the workshop.

The possibility of compromising the instructor role is also an issue. The dual role of a researcher and a teacher can enhance both roles, yet one must be aware that a teacher-researcher's first responsibility during class is to be a teacher (Tabach, 2011). During the sessions I was the instructor, and so focused on that aspect of my dual role. Practically, while I was teaching, I became involved in the lesson and mostly forgot the research aspect of the workshop session. However, I did find myself infrequently, subconsciously noting incidents that would be interesting to analyze. The researcher role protruded into the teacher role only in a fleeting manner. Immediately after the sessions were over, I recorded an audio journal entry about the session, and then switched to researcher role. The analysis of the data, which was done after class while in researcher mode, helped me be a better teacher. Critically engaging with one's own teaching practices supports the development of these practices by making the specific teaching goals more explicit (Jaworski, 1998). I am also more aware of student difficulties and possible issues that can arise in class due to watching and re-watching the recorded videos. Both roles - teacher and researcher - are enhanced by the other role.

4.5 Ethical Statement

This project was reviewed and approved by the Behavioral Sciences Research Ethics Committee of the Technion - Israel Institute of Technology (Certificate No. 2019-063). It had the approval of the Vice Dean of Undergraduate Studies of the Mathematics Department, the Head of the Technion International School of Engineering, and the courses' staff. The students signed an informed consent form. All names are pseudonyms and confidentiality of the students was preserved throughout the analysis and the writing.

5 Linear Algebra Tasks

In this section the tasks used as the basis for the workshops are examined. The workshops were designed to support explorative participation through discourse rich instruction. Thus, the tasks, which are at the heart of the workshops, must have the potential to spark a meaningful academic discussion and the potential to support explorative student participation. The potential of a task for explorative participation was operationally defined, in the theoretcial background, as the existence of the possibility of presenting different realizations for mathematical objects and the possibility for authoring links between these realizations. As different realizations and links between them can support meta-level learning, the tasks were also analyzed for evidence of the meta-level and object-level learning involved in solving them.

I first present the DMT tool developed and used to explore the mathematical objects that can be exposed through the tasks, the different realizations of these objects and the opportunities for saming that can be afforded by these tasks. The DMT tool, discussed in the methods section in detail, is a visual representation of the realizations of a mathematical object and the subdiscourses available to the students within which the object can be realized. This allowed me to examine the potential of the tasks for supporting a meaningful academic discussion with multiple realizations from within multiple subdiscourses.

Then I present a commognitive analysis of the discourses involved in solving these tasks to examine if, and how, these tasks could support meta level learning in a classroom. The links between the different realizations, displayed in the DMTs, and the transitions between the different subdiscourses involved in each task are considered to examine the potential for object-level learning and meta-level learning.

The findings are first exemplified in detail for a specific task, and then described more generally for the other tasks later in this section. Thus, the findings pertaining to the SLE (systems of linear equations) task is first presented. The tasks are named by the topic they were designed to be used for.

5.1 The objects that can be exposed through the SLE task, their different realizations, and the opportunities for saming

In this section the objects that can be exposed through the tasks, their different realizations, and the opportunities for saming were examined by constructing and examining a DMT for the SLE task. First the process of constructing a DMT for a mathematical task is detailed and then what objects that can be exposed through the tasks, their different realizations, and the opportunities for saming are described.

5.1.1 Construction of a DMT exemplified on the SLE task

This section describes the process used to construct a DMT for the following task.

Task: Give a system of linear equations whose solution is the set $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$.

5.1.1.1 Constructing a DMT Step 1: Determine the root node

The construction of a DMT starts by determining the root node, which is the mathematical object involved in the task. The task here is to determine what system will have the given set as its solution. Solving this task includes, mainly, the exploration of the system of linear equations (SLE) object, thus the mathematical object is the system of equations. The wording of the task, "Give an SLE…" also shows that this is the main object embedded in the task. The SLE object has many realizations, and theoretically, all realizations of an object are equivalent. Thus, any realization can be the root node. For convenience's sake, the title given to the object in textbooks was chosen as the title in the root node. Therefore, the root node of the DMT for the mathematical object of this task was chosen as "SLE".

5.1.1.2 Constructing a DMT Step 2: List possible realizations and types

In the next step of constructing the DMT, possible realizations are listed, and general types found. Below are realizations that can be used in solving the above task and realizing an SLE.

First, an SLE can be realized as a list of linear equations with variables. This realization is familiar to most secondary school students. For example:

$$\begin{cases} 2x - y = 0 \\ 3x - z = 0 \end{cases} \text{ or } \begin{cases} 2x \ = y \\ 5x = \ \text{y} + \text{ z} \end{cases}$$

The next realization is one to which students are exposed during the beginning of a linear algebra course, that of an augmented matrix. Specifically, for the SLE considered in this task, the matrix could look like this: $\left(\begin{matrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right)$.

An SLE can also be realized by a solution set, in this example $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$. These are the constraints on the solutions to the desired system. This realization does not uniquely characterize an SLE, as there are infinitely many SLEs with this solution set. However, they are equivalent in the sense that the Gaussian matrix representing these systems will have the same row space. This can be realized as a general element of a set: $\{(x,2x,3x) \mid x \in \mathbb{R}\}$, the linear span of a finite set: Span{(1,2,3), (4,8,12)}, or the kernel of a linear transformation: Ker (T(x,y,z) = (y-2x,z-3x)).

An SLE is a mathematical object, and thus can be realized by its *properties*. For example, the system consists of 2 equations with 3 variables, it is a homogenous system, and it has a system rank of 2. This type of realizations also does not uniquely characterize a unique SLE, but rather are for a family of SLEs. However, they do realize the desired SLE.

There are also properties of the SLE which pertain to the SLE's solution set which realize the SLE. These include properties such as the zero vector is a solution of the SLE, the solution of the SLE has one degree of freedom, and there is a single parameter in the solution set of the SLE. These realizations also are for a family of SLEs, and not a unique SLE, similar to the previous type of realizations.

An SLE and its solution set are realizations of the same mathematical object. They both give conditions on a set of vectors; however, the system of equations is the list of conditions, and the solution set is the vectors that fulfill those conditions. The difference can also be described as the solution set is explicitly a set of vectors with constraints, whereas the list of equations realizes the set of vectors that solve the system, but it is not stated, nor symbolized, explicitly that it is the set of solutions of the system of equations.

These realizations above were authored by an experienced instructor familiar and knowledgeable with the mathematical topics and notions involved. There are also realizations mentioned by students during the workshops. New realizations can keep being authored, however there is no need to give an exhaustive list of all the possible realizations. Since the ultimate goal of this process is determining the types of realizations, that is the subdiscourses involved or the branches in the tree, the list given needs to be sufficient for this purpose. If there are additional realizations and additional types of realizations, these can be added to the DMT. Therefore, the process of listing realizations has an end.

Additionally, I needed to determine which realizations are repetitions. For example, are the realizations x+y=3 and 2x+2y=6 equivalent, and thus the second one is redundant and should not be included? Are the realizations x+y=3 and 3=x+y equivalent? The answers to these questions depend on the audience's mathematical metarules. The answers would be different if an elementary student was asked, if a first semester university student was asked or if a mathematical researcher was asked. The realizations x+y=3 and 3=x+y would be considered the same using the metarule of commutativity which a mathematical researcher would have adopted. In contrast, a 6 year old would probably not yet have adopted this metarule. The DMTs constructed were based on the students participating in this project, namely first semester students. Thus, mathematical equivalencies like x+y=3 and 3=x+y are considered repetitive, since commutativity is a metarule usually adopted in pre-university school. In contrast, since systems of equations are new objects for the students, x+y=3 and 2x+2y=6 are considered disparate realizations.

5.1.1.3 *Constructing a DMT Step 3: Determining the branches of the DMT.*

In this step of constructing a DMT, realizations are placed in branches of the tree, each type of realization in its own branch. Thus, in this step a classification of the types of realizations is carried out. Each type of realization is a different subdiscourse, as can be seen from the different keywords, different procedures and narratives that can only be stated within that subdiscourse. For example, the realization $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right)$ uses matrices, rows and rank as keywords. The procedures include row reduction and determining rank. The narrative Rank (A) = Rank (A|b) cannot be stated in the subdiscourse of lists of equations. Thus, the realizations with matrices were all classified as in the subdiscourse of matrices.

Four types of realizations, or subdiscourse, for the mathematical object SLE were determined. These are lists of equations, matrix representation, properties of the SLE and the solution space. The solution space can be realized as a set or as a vector space, since the set of vectors $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$ is both a subset and a subspace of $\mathbb{R}^3$. Therefore, there are five types of realizations, or five branches in the tree. As described in the theoretical background, the choice of branches, or subdiscourses, is supported by the historical development of the representations of SLEs.

5.1.1.4 *Constructing a DMT Step 4: Drawing the DMT.*

This step is drawing the actual tree. The node chosen in Step 1 is the root of the tree. The branches of the tree are the types of realizations found in Step 3, which are the historical domains or the subdiscourses involved for the SLE object. The realizations are those listed during Step 2.

The realizations included in the branch of subspaces are presented to the students much further on in linear algebra courses and are not available to students first learning about SLEs. Expert mathematicians and students with prior knowledge would use these realizations, and students solving this task at a later point in the course would also be familiar with this type of realizations. This branch of the DMT is shaded grey to signify that this branch exists, but that it is not available to the students.

The DMT for the above task, which deals with the mathematical object SLE is shown in Figure 5-1, below.

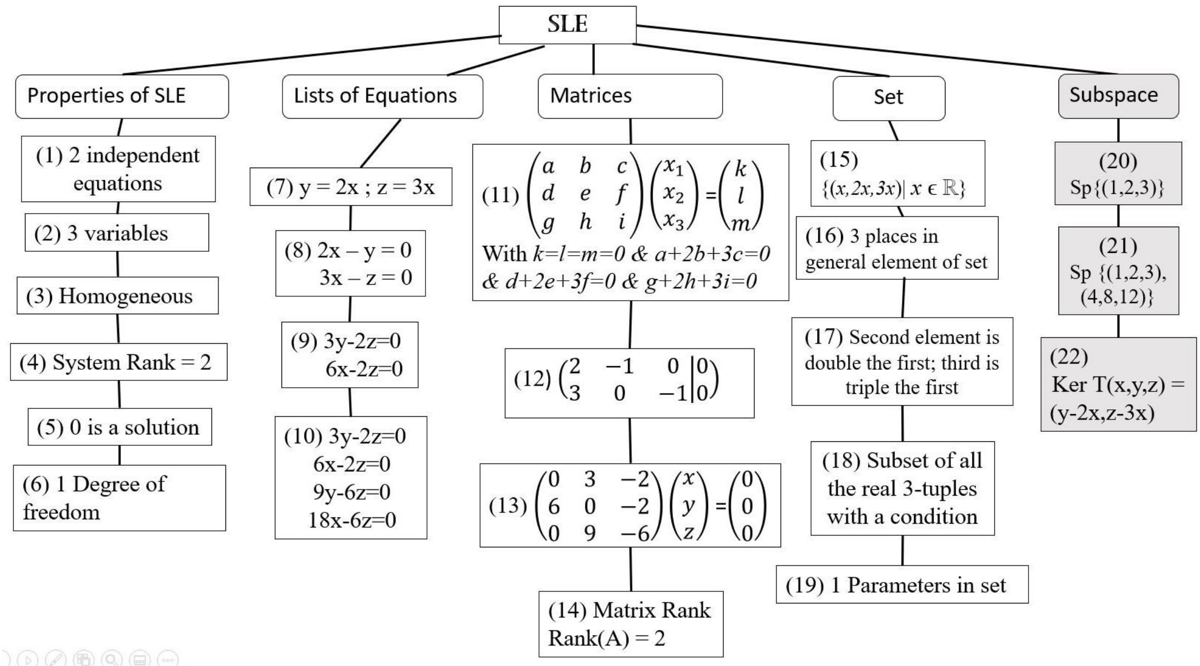

Figure 5-1 - DMT for SLE task

5.1.2 Objects that can be exposed through the SLE task, their different realizations, and the opportunities for saming

By examining the DMT displayed in Figure 5-1 the objects that can be exposed through the SLE task, their different realizations, and the opportunities for saming are apparent. The DMT demonstrates that embedded in the task there were at least four different types of realizations available to the students, as seen in the major branches stemming from the root object. That is, realizations in four different discourses are available in this task. This shows that there were multiple opportunities for students to same the various realizations of the mathematical objects.

The process of constructing DMTs allowed us to pinpoint the objects exposed by the task and the number of realizations for these objects that had been learned in the course. This offers a general view of the richness embedded in each task, that is, that in these tasks there exists the potential for multiple realizations and for constructing saming links between them. The multiplicity of these realizations demonstrates the potential of the task to support explorative participation,

5.1.3 Object-level learning and meta-level learning involved in the SLE task

In the previous section the DMT constructed for the SLE task displayed that the potential for multiple realizations in multiple discourses exists in this task. This also demonstrates the opportunity for both object-level learning and meta-level learning.

The DMT demonstrates the opportunity for object level learning, that is the opportunity for adopting new object level narratives in an endogenous development. The students are afforded the opportunity to author narratives within each of the subdiscourses displayed on the DMT. These can include authoring a realization within a subdiscourse or saming between two realizations within the same subdiscourse. For example, saming two mathematically equivalent lists of equations. The saming is within a single discourse and thus does not involve meta-level learning. This type of learning is also important for students, yet is more readily available in most standard tasks, and thus is not the focus of these tasks.

The DMT also demonstrated that the SLE task has opportunities for meta-level learning of adopting object related metarules in horizontal exogenous development. This includes the authoring of narratives in the coalesced discourse connecting between two subdiscourses. The DMT shows that the opportunity for unifying the different subdiscourses for each mathematical object in this task are available. The potential for constructing links between the branches of the DMT signals the potential for meta-level learning. For example, the narrative the rank of the matrix is 2, so the system of equations has 2 linearly independent equations. This narrative connects between a realization in the matrix subdiscourse (the rank of the matrix is 2) and a realization in the list of equations subdiscourse (has 2 linearly independent equations). Thus, this is a narrative in the coalesced discourse of SLEs and authoring this is meta-level learning.

The DMT displays the availability of multiple realizations in different discourses and thus shows that the SLE task affords opportunities for saming different realizations in different subdiscourses. Thus, the task has the potential to encourage and support objectification of the mathematical object SLE. However, the DMTs do not allow us to see the extent to which the task demanded the use of more than one (object-level) branch, only that the potential exists. For this, a more micro-level analysis of possible routines for solving the tasks is necessary.

5.2 A commognitive analysis of the SLE task

A commognitive micro-analysis of the discourses involved in solutions of this task were carried out. The solutions to the tasks were examined to determine if they could be obtained by following familiar routines from one discourse, or whether the solution necessitated following routines from different discourses.

First a possible solution was described to analyze the mathematical narratives necessary for a solution to the task. There are many possible solutions, this one was determined to include the necessary narratives of any solution by mathematical experts. This is explained in detail later. Next, the discourses involved in this solution were examined. Finally, the transitions between the discourses were considered. This is discussed in the following sections.

5.2.1 Possible solution for the SLE task

Following is one possible solution (out of many) for the SLE task.

Task: Give an SLE whose solution is the set $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$

The solution is presented below. The realizations used in the narratives are numbered to correspond to the node in the DMT of the task, in Figure 5-2 below the solution.

Solution:

(a) There are 3 places in the general element of the given set (16) that solves the SLE so there are 3 variables in the expected SLE (2).

(b) The general element of the set that would solve the expected SLE can be expressed using a single parameter (19), which is equivalent to stating that there is one degree of freedom in the expected SLE (6).

(c) The degrees of freedom of an SLE (6) is the number of variables (2) less the rank of the representative matrix (14), thus the rank of the system is 2 (4). That is, there are two independent equations in the expected SLE (1).

(d) The elements of the set that solves the SLE are the 3-tuples whose second element is double the first element and the third element is triple the first element (17), thus the conditions on the set can be expressed as y = 2x & z = 3x & x, y, z ϵ ℝ (hybrid between 7 & 18) or as 2x – y = 0 & 3x – z = 0 & x, y, z ϵ ℝ (hybrid between 8 & 18).
(e) The solution to the task is the SLE which is $\begin{cases} 2x - y = 0 \\ 3x - z = 0 \end{cases}$ (8) or $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right)$ (12).

In the figure below, Figure 5-2, the realizations mentioned in this solution are shaded grey. The branch of subspaces is faded out, as it was not available to the students at this point in the course. This demonstrates that the realizations included in this solution are from multiple discourses, as there are shaded boxes in each branch of the DMT.

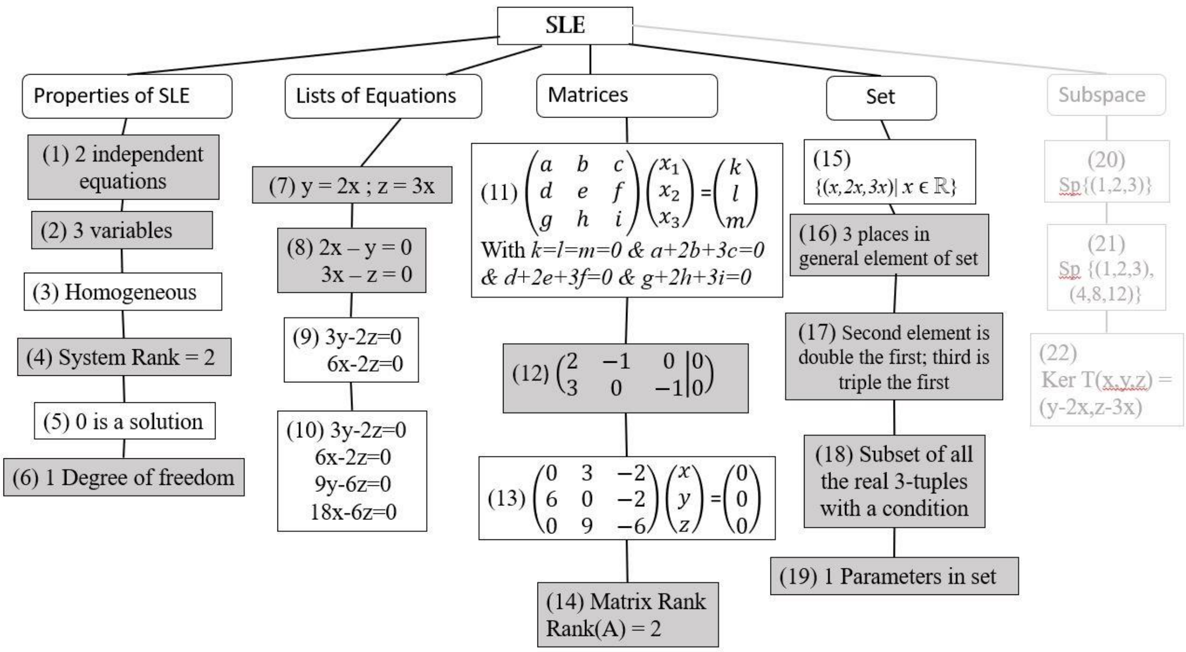

Figure 5-2 – DMT SLE possible solution

Although there are many possible different solutions to this task, there are certain narratives that must be included in any solution. First, the number of variables in the system must be determined. Although the narrative *there are n variables in the system* may be implied, and not stated out loud, it will be included in any solution when the suggested system is written out using n variables. Any solution must include writing down a realization of an SLE, and thus the number of variables or the number of columns in a matrix must be determined.

Similarly, the narrative *there are at least m equations in the desired system* must also be included. Any solution must determine how many equations to write down, and this must be determined somehow. Any solution written down will also include determining if the system if homogeneous or not (Ax = b is homogeneous if b=0). As with the other narratives, this might not be explicit but implied by the final solution. The solution described above includes these narratives and can be considered typical of any solution for the task in that respect.

I now present an alternative solution and show that these narratives are present. This solution is based on student unsuccessful attempts at a solution.

Additional Solution:

a) The equations in the requested SLE are of the form ax+by+cz=d.

b) The system is homogeneous, thus d = 0.

c) Plugging in the given solution, (x,2x,3x) yields:

   ax + b(2x) +c(3x)=0

d) Thus, a+2b+3c =0

e) Choose various a,b,c ∈ ℝ such that this holds.

f) $\begin{cases} x - \frac{z}{3} = 0 \\ y - \frac{2z}{3} = 0 \\ -3x + z = 0 \end{cases}$ is such an SLE

This solution includes the narrative *there are n variables in the system* in step (a). The determination of the structure of the desired equation includes the number of variables. Step (b) is the narrative *the system is homogeneous*. In step (e) of the solution the narrative *there are at least m equations in the desired system* is implied in the choice of how many different values to determine for a,b and c.

The original solution presented earlier, the solution presented here, and any other solution of this task includes some mandatory narratives. Therefore, the solution suggested above, which includes these aspects of any solution, was analyzed in detail to explore this task.

5.2.2 The discourses involved in solving the SLE task

The above solution was just one of many possible ways to solve the task. Nevertheless, examining more closely how it transitions between discourses demonstrates that this task encourages such transitions. To do so, the objects involved in each step of the solution were examined to determine which discourses are involved in solving this task.

There are two main mathematical objects of this task – a list of equations and the set $\{(x,2x,3x) \mid x \in \mathbb{R}\}$. The major meta-level task of this question is the saming between the discourse of SLEs and the discourse of sets. The objects from these, and other, discourses involved in the solution presented above are examined.

In the possible solution given in the previous section, the statements are labelled by letters, (i). In the first statement (a) “the number of places in the general element of a set”, an object from the Set discourse, is stated to be equivalent to “the number of variables in the SLE”, an object from the SLE discourse. The second statement (b) equates “the number of parameters", an object from the Sets discourse, with "the degrees of freedom", an object from the SLE discourse. Statement (c) contains objects from the SLE discourse – "degrees of freedom", "number of variables" and "number of independent equations" – and links to an object from the Matrix discourse – "rank(A)". These statements present objects from the discourses of Sets, SLEs and Matrices and states saming links between the various objects.

Statement (d) entails hybrid mathematical objects, that is objects that are constructed from two discourses. The conditions on the set are presented in the Sets discourse as "the second element is double the first element and the third element is triple the first element". These are then presented written algebraically as a list of equations, "$y = 2x$ & $z = 3x$ & $x, y, z \in \mathbb{R}$". The conditions on the set as a list of equations uses the Sets discourse and the List of Equations discourse. The final statement (e) recognizes these algebraic conditions from the List of Equations discourse as the SLE the task searched for.

This solution used four different types of discourse – Properties, Matrices, Lists of Equations and Sets. These are the different branches displayed in the DMT above.

The number of equations and variables in the expected system must be determined as part of the solution, and this includes narratives in the "properties of SLE" discourse. Degrees of freedom is a nascent vector space term that symbolizes the rank of a vector space, without any formal definitions. This hybrid construct is a scaffold used when SLEs are presented in the course before vector spaces. This order of topics allows SLEs to be used as an illustrative example of a vector space and supports intuitive understanding of the vector spaces. The SLE discourse is new to the students and is introduced and exemplified in the lectures and tutorials.

The Matrix discourse is used to discuss representative matrices of coefficients of SLEs, such as used in the Gaussian method of solving an SLE. This discourse includes matrices, augmented matrices, row reduction and echelon form. These objects are part of the topics introduced to the students in the course before SLEs are introduced. The efficiency of the matrix realization and the integral part it plays in the process of finding a solution to a system supports the students' adoption of it almost immediately and. exclusively once they are introduced to it. Thus, a matrix representation of an SLE can be considered an acceptable solution to the task. The matrix discourse is also necessary to determine if a suggested SLE is a solution to the task, by using matrix representation and the Gaussian method to solve the suggested system. The phrasing of the task necessitates the Sets discourse, and the final answer to the task is from the List of Equations discourse.

Examining this solution displayed that four different subdiscourses– Properties, Matrices, Lists of Equations and Sets – are involved in solving this task.

5.2.3 Transitions between discourses included in the solution

The presented solution included four different subdiscourses, as shown above. In this section the transitions between these discourses are examined. These transitions between discourses support the saming of realizations between the different discourses.

The task, as phrased, includes transitions between discourses. The beginning of the task, "Give an SLE whose solution is…", belongs to the List of Equations discourse involving "equations" and "solutions" of these equations. If a student employed the familiar routines of solving sets of equations, from high school or Gaussian elimination, they would reach an impasse, since those routines are appropriate for finding a solution of a given system. However, in this task a student must first construct a system. The last part of the task, namely "the *solution* is the *set* $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$", belongs to the Sets discourse. As the expected answer is a list of equations or a matrix representation, the Sets discourse is also not sufficient. Thus, any solution necessitates tapping multiple discourses and linking between them.

The task states that the solution is "the set $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$", from within the Sets discourse. Thus, solving this task begins in the Sets discourse. Possible routines available to students in this discourse could be phrased as answers to sub-tasks such as "what can we say about this set?", or "how would we characterize the elements of this set?" The narratives resulting from these sub-tasks would include, "For every real value x the 3-tuple $(x,2x,3x)$ is in the set" or "The set is a subset of all the 3-tuples that can be expressed using a single parameter". However, after authoring these narratives the students are liable to again reach a "dead-end". There are no available routines within the discourse of sets to continue with these narratives, especially not any that would lead them to saying anything about "an SLE" of which this set is "a solution". Thus, this task cannot be solved within the discourse of Sets.

One very familiar routine for obtaining a "solution" for an SLE is that of Gaussian elimination within the matrix discourse which entails representing a system as a matrix, reducing it to echelon form and utilizing the row-equivalent system to determine the solution space. This is the main, standard routine used in solving SLEs, thus the students would turn to this routine. However, in this task no system is given. There is nothing to "reduce". The students must construct their own system (where the solution is given) and the familiar routines for *finding* solutions are not helpful for that. This task cannot be solved using exclusively familiar routines from within the matrix discourse.

Another routine for solving the task would be to suggest random SLEs and examine the solution space of these, in an attempt to discover an appropriate SLE. This trial-and-error process, with no operational method of choosing equations from among infinite possibilities is not practical and leads to frustration at the enormity of the task. Thus, the familiar routines in the discourse of SLEs and Matrices are not sufficient and again lead to "dead-ends".

When students reach an impasse, and cannot continue, they can be guided to search in other discourses for a possible routine. Thus, the routines of "what can we say about this set" can be shifted to the discourse of SLEs, which includes the term "degrees of freedom". The new task can be "what can we say about this set in the SLE discourse" and can result in the narrative, "there is one degree of freedom in the set". This narrative can prompt a familiar routine in the discourse of SLEs/Matrices, which uses the endorsed narrative in this discourse: *degrees of freedom is equal to the number of variables less the rank of the system*. Thus, the transition between the discourse of Sets to the SLE discourse can be encouraged.

Similarly, the narrative "the conditions on the set can be expressed as $y = 2x$ & $z = 3x$ & $x, y, z \in \mathbb{R}$" which is within a hybrid of the discourse of Lists of Equations and the discourse of Sets brings the student to an impasse within the discourse. Completing the task by recognizing this as the SLE necessitates transitioning out of the discourse of Sets, and saming this set with an SLE.

Solving this task includes the narrative *the SLE has three variables since the general element of the given set of solutions has three places*. The first part of this narrative, *three variables*, is from within the discourse on properties of SLEs and is marked (2) in Figure 5-2. The second part, *the general element of the given set of solutions has three places,* is from within the discourse on sets and is marked as (16) in Figure 5-2. The narrative traverses two discourse and cannot be stated in either of the discourses singly. This narrative can only be stated in the new coalesced discourse of both the discourse of Sets and SLEs together.

This is displayed in Table 5-1, below. This table shows a possible subroutine for solving the task together with an analysis of the discourses traversed in each sub-routine. The narrative described above is labeled (a) and is the first row of the table. The nodes on the DMT (Figure 5-2) that represent each realization are marked in parentheses.

Routine sub-step	**Discourse traversed**
a) There are 3 places in the general element of the given set (16) that solves the SLE so there are 3 variables in the expected SLE (2).	Sets (general element) → Properties of SLE (3 variables)
(b) The general element of the set that would solve the expected SLE can be expressed using a single parameter (19), which is equivalent to stating that there is one degree of freedom in the expected SLE (6).	Sets (general element) → Properties of SLE (degree of freedom)
(c) The degrees of freedom of an SLE (6) is the number of variables (2) less the rank of the representative matrix (14), thus the rank of the system is 2 (4). That is, there are two independent equations in the expected SLE (1).	Properties of SLE (degree of freedom, number of variables) → Matrix discourse (rank of matrix) → SLE (two independent equations)
(d) The elements of the set that solves the SLE are the 3-tuples whose second element is double the first element and the third element is triple the first element (17), thus the conditions on the set can be expressed as y = 2x & z = 3x & x, y, z ϵ ℝ (hybrid between 7 & 18) or as 2x – y = 0 & 3x – z = 0 & x, y, z ϵ ℝ (hybrid between 8 & 18).	Sets (elements of the set, 3-tuples, etc.) → System of equations (e.g. y=2x)
(e) The solution to the task is the SLE which is $\begin{cases} 2x - y = 0 \\ 3x - z = 0 \end{cases}$ (8) or $\left(\begin{array}{ccc\|c} 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right)$ (12).	Systems of equations, matrices, SLEs.

Table 5-1 - Discourses traversed

As can be seen in Table 5-1, this routine tapped four different discourses – Properties of SLEs, Matrices, List of Equations and Sets and included transitions between them. The transitions in the above solution are necessitated by students following familiar object-level routines within a discourse and reaching an impasse, or a dead-end, when the object level routine does not continue. Thus, the task necessitates implementing meta-level routines of linking to routines and narratives in another discourse.

5.2.4 Summary of the commognitive analysis of the SLE task

In this section, a possible solution to the SLE task was presented, and the objects, routines and discourses involved in this solution were closely examined. Solutions for this task include realizations and narratives from multiple subdiscourses, transitions between these discourses and impasses when attempting to solve the task within a single discourse. Thus, this task afforded opportunities for practicing meta-rules involved in linear algebra by supporting and necessitating cross discourse narratives.

Importantly, the assumption underlying this task is that students *have already been* introduced to the various subsumed discourses underlying the SLE (full linear algebra) discourse, as well as to the equivalence (sameness) of the various realizations. Thus, this task is not supposed to introduce students to new meta-rules but rather to support them in enacting and rehearsing the saming actions that are critical for the objectification of SLEs. Still, given the difficulty of meta-level learning (Sfard, 2007b), it is conceivable that the instructor would have an important role in supporting students' struggle with such tasks. Thus, when students reach an impasse, the instructor should guide them to search in other discourses for a possible routine.

5.3 The objects that can be exposed through other tasks - their different realizations, and the opportunities for saming

In the previous two sections the findings for a single task were presented in detail. Tasks about other topics were designed for the workshops and these were examined. In this section the DMTs for these tasks are presented.

Similar to the detailed process described above for the construction of the DMT for the SLE task, DMTs were constructed for 6 other linear algebra tasks. This process included determining the root node, listing realizations, and determining the different subdiscourses available for each mathematical object to be used as branches of the DMT. In the following sections these are presented along with the DMT constructed.

In each section first the task is given. Then the choice of root node, that is the mathematical object for which the DMT was constructed, is briefly explained. The subdiscourses within which the object can be realized are discussed and exemplified. Finally, the DMT constructed using these is displayed. The section headings are the names given to the tasks. These names are based on the label of the topic as used in standard linear algebra textbooks and syllabi.

5.3.1 Complex Numbers Task

Task Let $z_1, z_2 \in \mathbb{C}$ such that $z_1, z_2 \neq 0$.

1) Let $z_1 \cdot z_2 \in \mathbb{R}$. Which of the following statements is always true? Which statement is never true? Which statement holds for specific cases of $z_1, z_2 \in \mathbb{C}$?
    a) $z_1 = \overline{z_2}$
    b) $z_1 = \alpha \cdot z_2 \, (\alpha \in \mathbb{R})$
    c) $z_1^2 \cdot z_2^2 = 1$
    d) $\mathrm{Im}\,(z_1) = 0$

2) Give a statement for which the following is true: $\frac{z_1}{z_2} \in R \Leftrightarrow$ (statement)

Node The mathematical object used as the node in this DMT is the complex number, as a general element of $\mathbb{C}$.

Discourses A complex number can be realized in the algebraic discourse, for example 3+4i. This discourse includes narratives about the real part of a complex number (Re(3+4i)=3), the imaginary part of the complex number (Im(z) = 4) and the modulus of a complex number (|3+4i| $=\sqrt{3^2+4^2}$). This discourse includes algebraic manipulation of real numbers.

A complex number can also be realized within the geometric discourse, where the complex numbers are realized as a dot on a 2-dimensional axis or on a plane. This discourse includes narratives about distance from the origin, geometric characteristics of right-angled triangles and quadrants of the plane.

The polar coordinate system discourse, also known as the trigonometric representation, can also be used to realize complex numbers. In this discourse the complex number can be represented using degree as 5(cos 53.13 + i sin 53.13) = 5cis (53.13) or using radians as 5cis ($\frac{\pi}{3}$). The narratives in this discourse include trigonometric functions and trigonometric identities.

There is also the real plane discourse, where complex numbers can be realized as a 2-tuple such as (3,4). This discourse includes narratives within $\mathbb{R}^2$.

Finally, complex numbers can be realized as roots of a polynomial. In this discourse there are narratives pertaining to polynomials, such as the first fundamental theorem of algebra which states that any polynomial of degree n has n complex roots (including multiplicities), and narratives about finding the roots of polynomials, such as if z is a root of a polynomial with real coefficients, then $\overline{z}$ is also a root.

The complex number object can be realized in five subdiscourses – algebraic, geometric, polar, planar, and polynomial. These are the branches of the DMT, shown below.

DMT

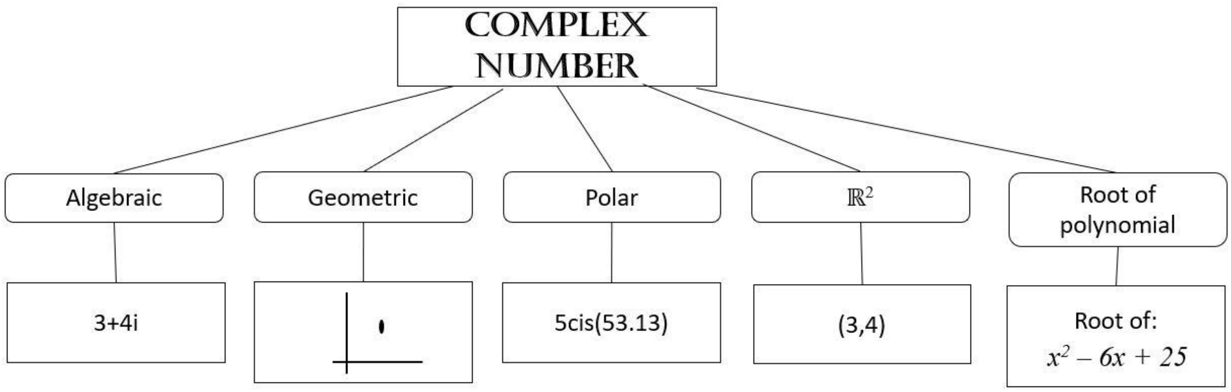

Figure 5-3 DMT Complex numbers

5.3.2 Matrices Task

Task Let C be a matrix whose third column is all zero's. Let D be a matrix whose second row is all zeros. Examine CD and DC. Do they inherit any characteristics from C and D? That is, is the third column all zeros? Is the second row all zeros?

Node This task discusses properties of the structure of matrices, such as a certain row being all zero, and multiplying matrices. Solving the task involves focusing on a matrix with a certain property and authoring narratives pertaining to it. A different matrix with another property would have the same branches, but the realizations would be different. The properties depend on which field is used to construct the matrices, but the properties of the field are not the focus of this task. Thus, the mathematical object was considered as a general matrix over a general field F.

Discourses Matrices can be realized visually as a block, where the objects being manipulated are a two-dimensional array, such as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This is a discourse including narratives manipulating the arrays as a single object, for example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This discourse also includes describing properties of the array, such as *the matrix is symmetric* (graphically, i.e. invariant under reflection with respect to the main diagonal) or *the matrix is square*. Matrices, and their properties, can be realized in this discourse using "hand motions". For example, a drawing with lines in large parenthesis realizes a matrix, even though no specific elements are apparent, as in the following image.

Each matrix can also be realized as an array of scalars, where each element in the matrix is a scalar and treated as its own object. This discourse includes narratives from within the field of scalars, such as *the element in the third row and second column is zero* (*a_{32}=0)* or *for all i and j it holds that* $A_{ij}=A_{ji}$. This subdiscourse includes familiar narratives about scalars, as the fields are familiar to the students, however realizing a matrix as an array of scalars is not simple.

A matrix is also an array of n-tuples, that is each row in a mxn matrix is an n-tuple and each column is an m-tuple. This discourse includes narratives such as *row k is equal to column k* and *the row space of the matrix is equal to F^n*.

Once matrices are mathematical objects they can be realized symbolically and by their properties as elements of the set of all matrices. For example, *A commutes multiplicatively on the left* or *multiplying by A on the right does not alter any matrix*. The symbol A realizes the matrix, as in the narrative A^t=A. This discourse realizes matrices by describing properties of them, rather than the internal structure.

Matrices can be realized in four subdiscourses – visually, arrays of scalars, arrays of tuples, and symbolically. These are the branches of the DMT, shown below.

DMT

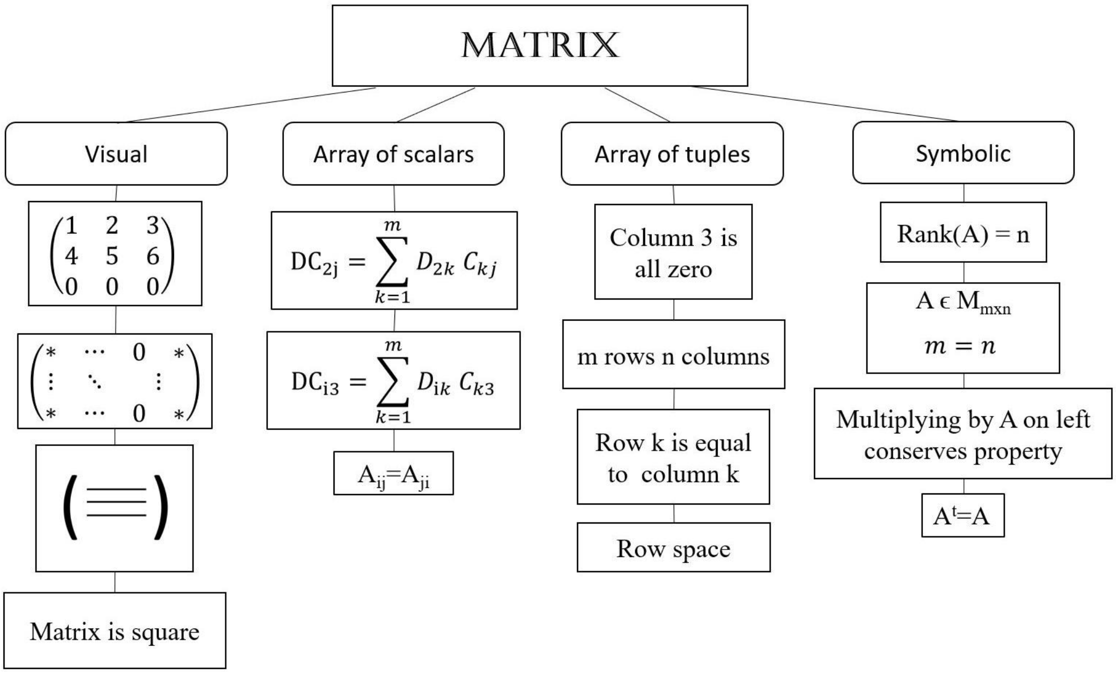

Figure 5-4 DMT Matrix

5.3.3 Vector Space Task

Task What is the greatest value of n, such that there exist subspaces $W_i \subseteq \mathbb{R}^{(2\times3)}$, $1 \leq i \leq n$, such that $W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{(n-1)} \subsetneq W_n$?

Node This task explores subspaces and the relationships between different subspaces of a given, real vector space. The scalars being real numbers does not affect the solution, therefore the object is not the real subspaces, but a general vector space.

Discourses A vector space can be realized as a set of elements. This discourse includes narratives such as *(0,0,0) is in W_2* and *W_1 is a subset of W_2*. In this discourse the elements are realized as n-tuples, that is a sequence of numbers separated by commas. The unique properties of vector spaces are not part of the narratives. Although these sets can be infinite (when the field is of characteristic zero), the narrative *x is an element of W* is within this discourse.

Another way in which vector spaces can be realized is as a set of vectors, where the relationship between the elements is noted. This discourse includes narratives such as *all the multiples of (1,0,0) are in the vector space*. Vector spaces can be realized in this discourse as linear spans, which include narratives such as *the basis of the vector space is (1,0,0)* or manipulation of the basis instead of the entire vector space.

Finally, as vector spaces are mathematical objects, they can be realized by their properties. This discourse includes narratives such as *the dimension of the vector space is 2* and *the vector space is a subspace of a different vector space*.

The object vector space can be realized in three subdiscourses – sets of elements, sets of vectors, and by its properties. These are the branches of the DMT, shown below.

DMT

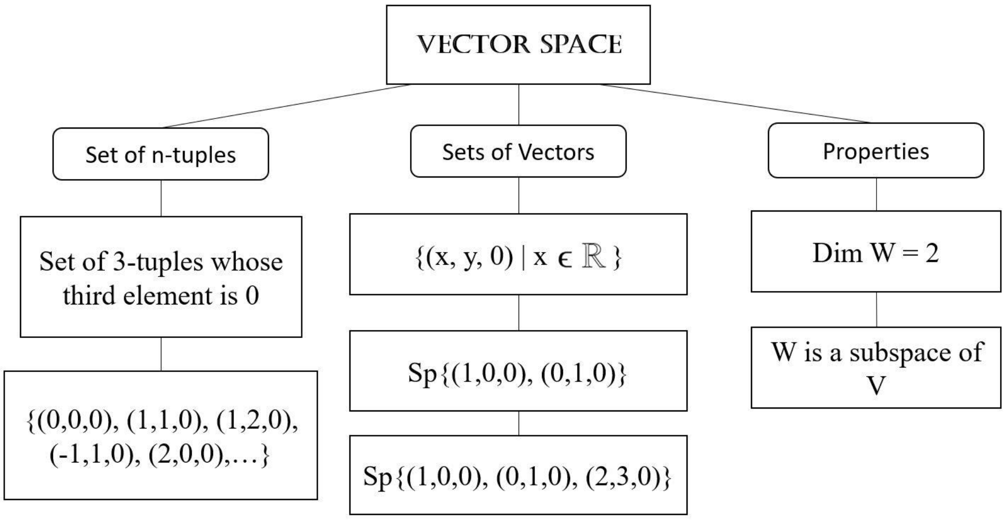

Figure 5-5 Vector Space DMT

5.3.4 Linear Dependence Task

Task V is a vector space over the field F. If a statement is true, prove it. If a statement is false, give a numerical counter example.

(1) Given the set $\{u_1, u_2, u_3\}$ is linearly independent and $u_4 \in V$, then the set $\{u_1, \ldots, u_4\}$ is linearly independent.

(2) Given the set $\{u_1, u_2, u_3\}$ is linearly dependent and $u_4 \in V$, then the set $\{u_1, \ldots, u_4\}$ is linearly dependent.

(3) If $\{u_1,\ldots,u_6\}$ is a linearly dependent set, then $Sp\{u_1,\ldots,u_5\} = Sp\ \{u_2,\ldots,u_6\}$.

(4) If $\{u_1,\ldots,u_6\}$ is a linearly independent set, then $Sp\{u_1,\ldots,u_5\} = Sp\ \{u_2,\ldots,u_6\}$.

Node This task includes sets of vectors that are linearly dependent and linearly independent. Solving this task includes authoring narratives about these sets. There is no specific set, rather it is a general set. The number of elements in the set is not central to the solution, rather the relation between the dimension of the vector space and the size of the set is important.

The notion of linearly independent vectors is logically equivalent to the notion of not linearly dependent vectors. These two objects are two faces of the same object and the DMT constructed for them is similar and use negations of the same narratives within the realizations. Therefore, either can be chosen as nodes and I used linearly independent vectors.

Discourses A linearly dependent set can be realized as a collection of n-tuples for whom there exists a linear combination equaling zero, that is as a collection of vectors. This discourse includes narratives about scalars and linear combinations, such as *there exist scalars* α, β, γ, *not all zero, such that* $\alpha \cdot v_1 + \beta \cdot v_2 + \gamma \cdot v_3 = \vec{0}$.

A linearly dependent set can be realized as a set of n-tuples with properties. This discourse can include narratives such as *the vectors in the set are multiples of each other* and *zero is an element of the set*.

Linearly dependent sets can also be realized by their representations as coordinate vectors in F^n. This discourse includes reduced echelon matrix representation of these vectors as narratives. For example, t*he row reduced matrix* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *does not have a row of zeros.*

The discourse about vector spaces can also be used to realize a linearly dependent set. For example, the narrative *(1,2,3) is not an element of the linear span of (1,0,0*) and four vectors cannot be linearly independent in $\mathbb{R}^3$ are within this discourse.

The mathematical object a linearly independent set can be realized in four subdiscourses – scalars, sets of n-tuples, matrices and vectors. These are the branches of the DMT, shown below.

DMT

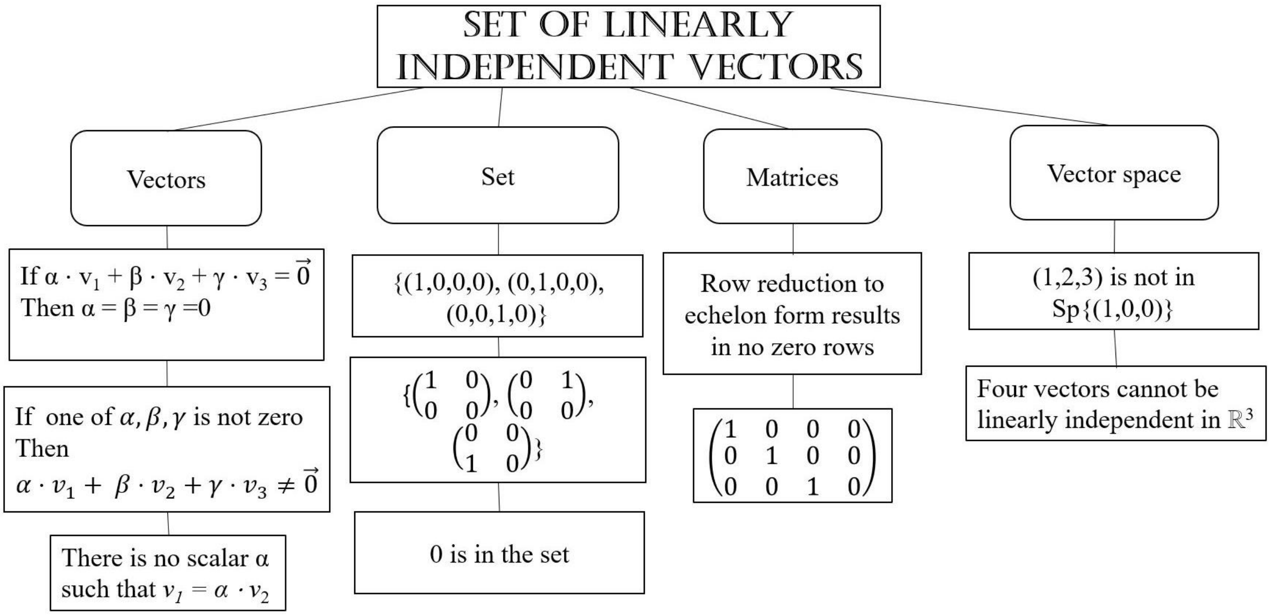

Figure 5-6 Linearly Independent Vectors DMT

5.3.5 Linear Transformation Tasks

The task used for the workshop implemented in the Algebra 1m course was considerably modified for the next iteration of the workshops in the Algebra A course, which included a lot more theory and finite fields. Thus, both tasks are presented.

Task (Algebra 1m) T: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation such that T(1,2,3) = (0,0,0) and T is not the zero transformation.

For which values of n $\in \mathbb{N}$ does there exist such a T so that dim Ker T = n?

For those values of n give an example of such a T and find a basis of Ker T.

Task (Algebra A) T: $\mathbb{Z}_5^4 \rightarrow \mathbb{Z}_5^4$ is a linear transformation such that T(1,2,3,4) = (0,0,0,0).

1) For which values of n $\in \mathbb{N}$ does there exist such a T so that dim Ker T = n?
For those values of n give an example of such a T and find a basis of Ker T.

2) If, in addition, there exist 3 vectors v_1, v_2, v_3 such that T(v_1) = T(v_2) = T(v_3), which values can dim Ker T take?
3) Construct a T that fulfills the given conditions and also Ker T = Im T.
4) Construct a T that fulfills the given conditions and also Ker T ∩ Im T = sp{(1,2,3,4)}.

Node These tasks include defining a linear transformation that has certain properties. Solving these tasks includes realizing the mathematical object of a linear transformation, thus this is the root node of the DMT.

Discourses A linear transformation is an expansion of a function, thus it can be realized in the discourse of functions. This includes narratives about the image of vectors, such as T(1,2,3) = (3,3,3), *the image of a vector (x,y,z) is (x+y,x+y,x+y)* and *the linear transformation is injective*.

A linear transformation can be realized using vector spaces. A linear transformation can be realized by its definition on any basis. The discourse of basis of vector spaces includes narratives such as *the linear transformation is uniquely determined by defining it on a basis*. Linear transformations can also be realized by the subspaces associated with them – the kernel and the image. Within this discourse the *narrative the image of T is the linear span of the vector (1,1,1)* can be stated.

A linear transformation can also be realized by its matrix representation according to a basis. The discourse of matrices includes narratives such as *the linear transformation is invertible since the matrix is invertible* and *the image of the linear transformation is the column space of the matrix*.

Linear transformations can also be realized as elements of a vector space, namely Hom(V,W). This extremely abstract notion was not included in all the algebra courses, and thus was not available to the students in all the workshops. It is not displayed on the DMT.

Thus, in this course the mathematical object linear transformation can be realized in three subdiscourses – functions, vector spaces, and matrices. These are the branches of the DMT, shown below.

DMT

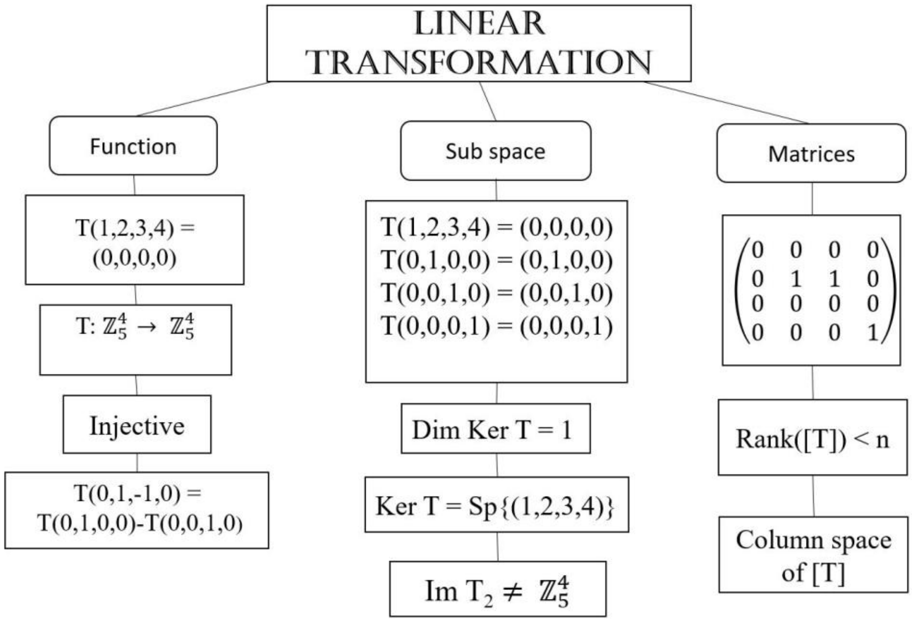

Figure 5-7 DMT Linear transformations

5.3.6 Diagonalizable Matrix Task

Task Let A be an n x n complex matrix, $A \in \mathbb{C}^{n \times n}$. $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$.

For what conditions on $a_1, a_2, \ldots a_n$ is A **not** diagonalizable?

Node This task discusses the mathematical object of a diagonalizable matrix. Yet, solving this task contains narratives about the mathematical objects eigenvalues and eigenvectors. These two topics are often presented to students together, as they are intertwined. The solution of this task demands a conclusion about a matrix and uses narratives about eigenvalues to justify these. Thus, the root of the DMT is the diagonalizable matrix, and the mathematical object of eigenvalue is a subtree of the main object.

In all trees there are potential subtrees. For example, the DMT for a complex number includes the mathematical object of a real number, which can be mapped by an DMT. The real number's DMT can, in turn, include a subtree mapping a rational number. This process will continue, expanding and lengthening the DMT to an unwieldy entity. Thus, the underlying assumption included in constructing DMTs is that although every object has a subtree, if it is an object that the learners are familiar with, no subtree is mapped for such an object. Therefore, the endpoints of DMTs are objects that can be considered as a prerequisite (familiar objects and discourse) for learning the new object. The DMT for diagonalizable matrices includes the subtree of matrices described above but is considered as a familiar discourse and is not mapped here.

Similarly, based on this assumption, eigenvalues should be considered as familiar, old objects within the diagonalizable matrix DMT. Yet, this is not the case. In the courses studied for this project, the two objects were taught together during the last week of the semester. Due to time constraints, the students were not able to first objectify eigenvalues and then examine diagonalizable matrices. Rather, it was all together. Thus, eigenvalues and eigenvectors were not familiar objects to the students and are mapped as a subtree within the main DMT.

Discourses Diagonalizable matrices can be realized as elements of the vector space F^{nxn}. This discourse includes narratives such as *the matrix is similar to a diagonal matrix* and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ *is a diagonal matrix*. This discourse includes realizing matrices as matrix representation of linear transformations.

The discourse of the vector space F^n can also be used to realize a diagonalizable matrix. For example, the narrative *the eigenvectors are a basis for the vector space* F^n is within this discourse. This discourse includes the object of eigenvectors. The discourses for this object are the same as for any vectors and are discussed in the task pertaining to vector spaces.

Diagonalizable matrices can also be realized as an array of numbers and by their scalar properties. For example, in this discourse the narrative *the eigenvalues are all different* can be stated and *the trace of the matrix is real* is also a narrative within this discourse. This discourse includes the subtree of eigenvalues, which is a scalar object that is central to solving this task, as explained above.

Eigenvalues can be realized in several discourses. They can be realized in the discourse of vectors. For example, *there exists a vector* $v \neq 0$ *such that* $A \cdot v = \lambda \cdot v$ is a narrative within this discourse. Procedures of vectors can be used on the eigenvectors.

The mathematical object eigenvalue can also be realized as the root of a polynomial. This discourse includes narratives such as *the algebraic multiplicity of the eigenvalue is the multiplicity of the eigenvalue as root of the characteristic polynomial* and *if* $\lambda \in \mathbb{C}$ *is a root of a real characteristic polynomial, then cong(*λ*) is also a root*. This discourse uses properties of scalars and polynomials.

Eigenvalues can also be realized within the discourse of matrices, as they are a property of matrices. For example, the narrative *0 is an eigenvalue of all non-invertible matrices* is within the matrix discourse.

The mathematical object diagonalizable matrix can be realized in three subdiscourses – matrices as vector spaces, vector spaces of n-tuples and by matrices with their scalar properties. The mathematical object eigenvalue can be realized in three subdiscourses – vectors, polynomials, and matrices. These are the branches of the DMT, shown below.

DMT

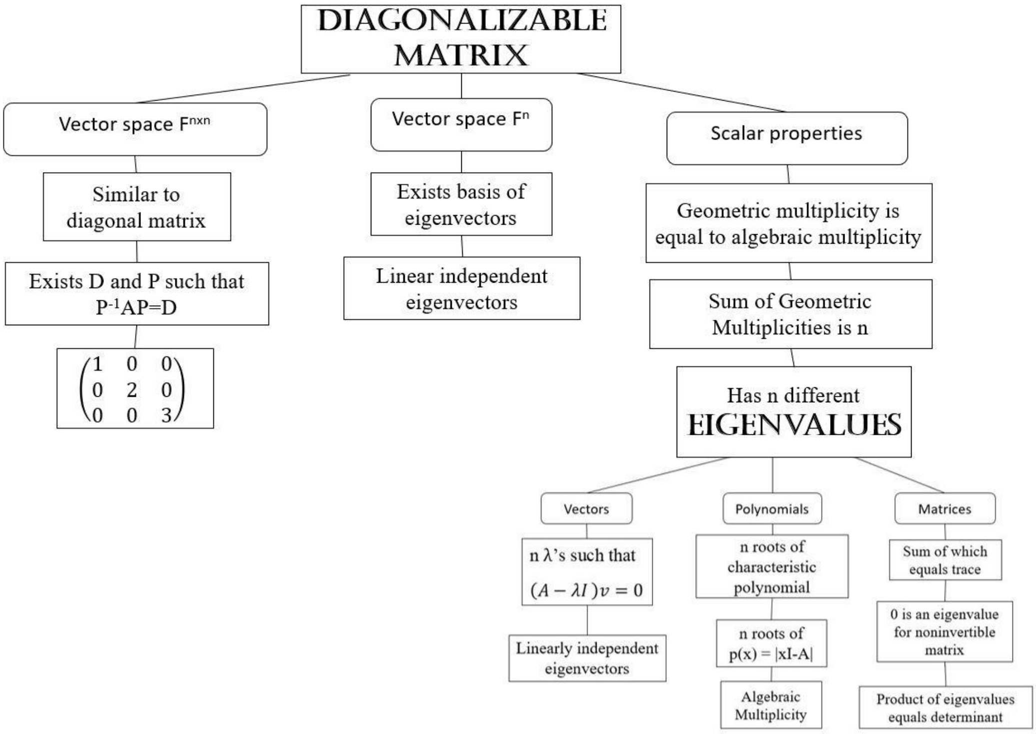

Figure 5-8 DMT Diagonalizable Matrices

5.4 Summary of chapter

The DMTs of all the tasks demonstrated that these tasks have the potential for authoring multiple realizations in multiple discourses, for constructing links between discourses and for transitioning between multiple discourses.

The commognitive analysis of the SLE task showed how solving this task encouraged these transitions. This task includes impasses, where the student has no available routines to continue within the discourse. At these nodes of the solution, there were other readily available routines in different discourses. There were opportunities for bonding between the final step end of a subroutine in one discourse and the first step in another discourse. Thus, this task encouraged transitioning between discourses and supported saming of realizations. Similar analysis was done on the other tasks and showed that such impasses that necessitated traversing between discourses were common in all the tasks. That is, these tasks have the potential to encourage explorative participation and include both object-level and meta-level learning.

6 Constructing DDMTs based on implementations of the tasks in linear algebra workshops

The previous section examined the potential of tasks designed to support explorative participation and encourage student learning. The potential of a mathematical task was defined as the capacity to provoke discussion, including compelling students to author realizations and links and providing the teacher with opportunities for highlighting unfamiliar links. Operationally, this potential was determined as the inclusion of multiple realizations and the athouring of links between these. As described in the methods section and the previous section, DMTs were constructed for six linear algebra tasks to examine the potential of these tasks.

The tasks afforded opportunities for both object-level learning and meta-level learning. At the object level, they afforded opportunities for authoring narratives within multiple subdiscourses. At the meta-level, they offered opportunities for authoring narratives in the coalesced discourse. This section looks at the implementations of these tasks to examine were the opportunities for explorative participation taken up and in what ways.

Explorative participation, as described in the theoretical background, consists of authoring object-level narratives in multiple discourses and connecting realizations from within separate subdiscourses. These connections are constructed by authoring narratives in the new, coalesced discourse. Thus, my goal for mapping the lessons was to examine if there were realizations from within different subdiscourses and were connections between these subdiscourses authored.

I mapped the realizations authored during the whole classroom discussion on a Discussion Discourse Mapping Tree (DDMT), which is a utilization of the DMT. The construction of the DDMTs, based on an actual discussion, includes both a priori and a posteriori components. The branches of the DDMT were drawn a priori, using the branches from the DMT. The branches are the available subdiscourses within which object-level narratives can be authored about this object. The drawing of the realizations on the branches was done a posteriori and based on what was mentioned in class. The construction of the DDMT supported mapping which subdiscourses were mentioned, which connections between subdiscourses were authored and who authored these.

This section first exemplifies the construction of a DDMT for a whole class discussion in a single workshop and then presents DDMTs from other workshops and what can be construed from these images.

6.1 An example of constructing a DDMT based on the recording of a whole class discussion in a workshop

6.1.1 DDMT for Workshop W5

This section displays the process of constructing a DDMT for workshop W5. The DMT for the task discussed in the workshop is in Section 5.3.1.5. This workshop was chosen to exemplify the construction process since it represents a typical workshop. It included multiple realizations and links, but not all the possible ones. Thus, the mapping described below can be considered as the process used for constructing all the DDMTs from the workshops. This workshop took place in the second half of the Winter 2020 semester. The students were from the Algebra A course. The discussions were held in Hebrew.

The workshop was based on the following task:

Task: T: $\mathbb{Z}_5^4 \rightarrow \mathbb{Z}_5^4$ is a linear transformation such that T(1,2,3,4) = (0,0,0,0).

1) For which values of n $\in \mathbb{N}$ does there exist such a T so that dim Ker T = n?
For those values of n give an example of such a T and find a basis of Ker T.

2) If, in addition, there exist 3 vectors v_1, v_2, v_3 such that T(v_1) = T(v_2) = T(v_3), which values can dim Ker T take?
3) Construct a T that fulfills the given conditions and also Ker T = Im T.
4) Construct a T that fulfills the given conditions and also Ker T ∩ Im T = sp{(1,2,3,4)}.

The workshop started with 7 minutes of a reminder of the basic theorems and definitions pertaining to linear transformations. Some of the definitions were written on the board by me, such as the linearity property of the transformation ($\forall\ \alpha \in F, \forall u, v\ \in V$ it holds that $T(\alpha u + v) = \ \alpha T(u) + T(v)$). Some were authored by the students after prompting. For example, I asked the students what the image of the zero vector is, and they answered it is the zero vector. Another example is that the students dictated to me the second dimension theorem (dim Im T + dim Ker T = dim V), which I wrote on the board. Some of the properties were in the subdiscourse of functions and some were in the subdiscourse of vector spaces. The linearity property can be stated in the subdiscourse of functions and the second dimension theorem, which pertains to the dimension of subspaces, uses the subdiscourse of vector spaces. This introduction reminded all the students of narratives in multiple subdiscourses. The students were familiar with these narratives and the subdiscourses were available to them for solving the tasks.

After the launch of the task, the students worked on the task in pairs for 15 minutes. This was followed by a whole class discussion that was 21 minutes long. The analysis herein focuses solely on the whole class discussion part.

The discussion about the first question of the task was rich and long, therefore the solutions to the other questions were only mentioned with minimal discussion. There were seven students in the classroom, and they all participated in the discussion. Some talked from their seats, and some came to the board to write out examples or to point to examples already written.

The construction of the DDMT commences by deriving its node and its branches from the DMT. The realizations drawn on the DMT used to determine these are not used. That is, the DDMT starts with an empty, labelled tree. This can be seen in Figure 6-1, below.

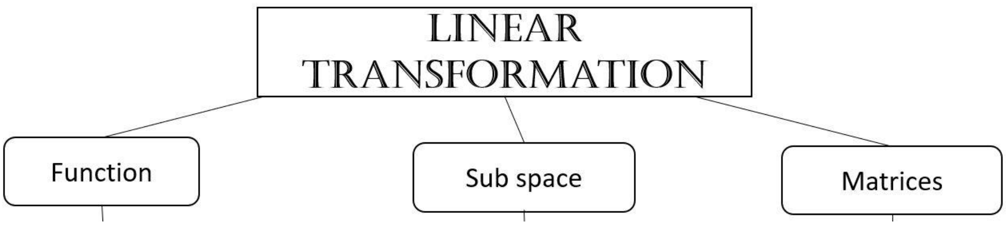

Figure 6-1: Blank DDMT for Workshop W5

Based on the recording of the whole class discussion, realizations that were mentioned were drawn on the appropriate branch and connections between realizations that were mentioned were also drawn. Dark boxes and solid lines were authored by students and light boxes and broken lines were authored by me, the instructor. The DDMT is below (see Figure 6-2) and is followed by a description of the mapping process.

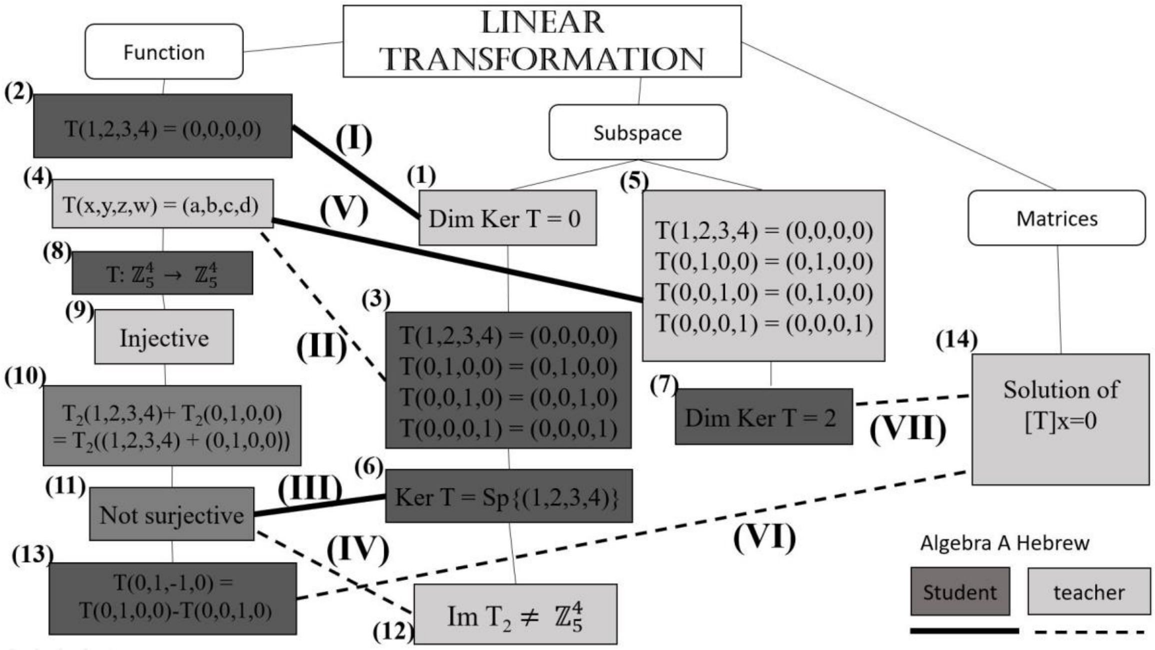

Figure 6-2 - DDMT Workshop W5

6.1.2 The mapping of Workshop W5

The discussion started by me asking does there exist a linear transformation such that dim Ker T = 0. This is a realization of T in the subdiscourse of subspaces authored by the instructor (me), therefore it was drawn on the subspaces branch of the DDMT in light grey and is labelled (1). The students answered “no” and justified their answer by explaining that there is a vector in the kernel, since T(1,2,3,4) = (0,0,0,0). This narrative, authored by the students, is in the subdiscourse of functions, as it claims that the image of (1,2,3,4) is zero, so the box labelled (2), in dark grey, was drawn. The students also connected between these two narratives and authored a narrative in the coalesced discourse, that if T(1,2,3,4)=0 then the dimension of the subspace, Ker T, is greater than zero. This connection is labelled (I) on the DDMT.

During the small group discussion, I noted that the students had constructed various examples for different values of n. Thus, I next asked for someone to give their example for n=1. A student volunteered to come to the board and said, “We can define the linear transformation by its behavior on the basis” and wrote as column vectors four vectors, which he later labelled a_1, a_2, a_3, and a_4, on the board. After justifying that {a_1, a_2, a_3, a_4} was indeed a basis, the student said that we can choose T(1,2,3,4) = (0,0,0,0) to fulfil the condition given in the question. He next explained that in order that dim kerT =1 the only other condition necessary is that the image of the other vectors be anything except for 0. He wrote:

$$\overset{a_1}{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}\ \overset{a_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}\overset{a_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}\overset{a_4}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}\qquad T(a_1)=0\quad T(a_2)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ T(a_3)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}T(a_4)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Figure 6-3 - Reconstruction of writing on board 1

This is a realization in the subspace subdiscourse and so box 3 was drawn in dark grey. For the sake of this discussion, I call this linear transformation T_1.

I mentioned that the final answer would need to be given in the general case, so the realization labelled (4) in the subdiscourse of functions was drawn. I asked if they all know how to do that and some students acknowledged this, so a connection was made between these two realizations. This was authored by me, the instructor, and so was marked as a broken line and is labelled (II) in the DDMT.

After asking the class what the dimension of the kernel of T_1 is, I changed the example on the board so that $T(a_2) = T(a_3)$.

$$\overset{a_1}{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}\ \overset{a_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}\overset{a_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}\overset{a_4}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}\qquad T(a_1)=0\quad T(a_2)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ T(a_3)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}T(a_4)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Figure 6-4- Reconstruction of writing on board 2

I claimed that this transformation, which for this discussion I call T_2, also fulfils the student's condition (that the image of all the other three vectors in the basis are not zero) and asked the class what the dimension of the kernel of T_2 is. Since I authored this realization of a linear transformation box (5) was drawn in light grey in the DDMT. The students expressed hesitation, so I asked, what is the kernel of T_1? A student said the span of (1,2,3,4), and thus box (6) in the subspace subdiscourse was drawn. Then I asked what is the kernel of the linear transformation T_2? A student said, "now its dimension is 2" and so box 7 was drawn.

A different student then asked, "is what we defined even a linear transformation?" This question led to a discussion of the existence and uniqueness of a linear transformation defined on the basis. First, one student claimed that the definition of T_2 such that T(0,1,0,0) = T(0,0,1,0) = (0,1,0,0) contradicts the definition of T_2 as a function. The realization (8) in the function subdiscourse was drawn. I explained the difference between well-defined and injective, and so box (9) was drawn.

Next, as part of proving that a linear transformation defined on the basis is well defined, a student showed that $T_2(1,2,3,4) + T_2(0,1,0,0) = T_2((1,2,3,4) + (0,1,0,0))$ on the board. This is one of the basic, defining properties of linear transformations and utilizes a realization of linear transformation as a function, and so box (10) was drawn.

Another student said that since for T_2 the set is not spanning the vector space, as opposed to T_1 which does, therefore T_2 is not well defined. His explanation of his question and my restating of it elicited from other students that T_2 is not surjective due to the kernel being not zero, and thus box (11) was drawn, and connection (III) was marked in a solid line. I clarified that the definition of surjective is that Im T = V and that $\text{Im } T_2 \neq \mathbb{Z}_5^4$, and in the case of a linear operator it is sufficient to say that $\text{Ker } T \neq \{0\}$. Box (12) was drawn, and line (IV) was marked.

A student suggested to find the general case of the transformation T_2 and so the connection between the definition on the basis in the subdiscourse of basis was connected by a student to the general case in the function subdiscourse, labelled (V). This was written out on the board with the students telling me what to write. Therefore, this was considered as a student-authored link. To conclude this part of the discussion, the theorem of existence and uniqueness of a linear transformation defined on a basis was mentioned and described.

I went back to the definition of T_2 and asked the class what is dim Ker T_2. There was some discussion about the dimension theorem (dim Ker T + dim Im T = dim V) and its corollaries if T is an operator. A student asked which two vectors can be in the kernel and another student suggested that the vector a_2-a_3 = (0,1,-1,0) is in the kernel, since $T_2(a_2-a_3) = T_2(a_2) - T_2(a_3) = 0$. The student authored the realization T(0,1,-1,0) = T(0,1,0,0)-T(0,0,1,0), which was labelled (13). I explained to the class that this is similar to the difference between two solutions of a non-homogeneous system of linear equations, which is a solution of the connected homogeneous system. Thus, a realization in the matrix subdiscourse was authored, labelled (14), and a connection labelled (VI) was noted. I mentioned that the solution of the homogenous system is the kernel of the transformation, and thus connection (VII) was made between the matrix representation of the homogenous system and the kernel.

6.1.3 Summary - the mapping of Workshop W5

To conclude, there are characteristics of the implementation of the task that are discernable from the image of the discussion as portrayed on the DDMT. There were seven realizations mentioned from within the subdiscourse of functions (boxes 2, 4, 8, 9, 10, 11, and 13). Six realizations were mentioned in the subdiscourse of subspaces, including narratives of subspaces as sets (boxes 1,6,7, and 12) and narratives of subspaces as vectors (boxes 3 and 5). Additionally, a narrative from the subdiscourse of matrices was authored (box 14). There were multiple realizations authored during the discussion, both by the students and by the instructor. There were also links authored during the discussion. Five connections were authored between the subdiscourse of functions and the subdiscourse of subspaces, two authored by the instructor (links II and IV), and three of them were authored by the students (links I, III, and V). One connection was authored between the subdiscourse of functions and the subdiscourse of matrices (link VI) and one connection was authored between the subdiscourse of subspaces and the subdiscourse of matrices (link VII).

The class authored realizations in multiple discourses and linked between them, that is they participated exploratively in the discussion. The class also practiced procedures from within the different subdiscourses and narratives from the coalesced discourse of linear transformations were encouraged, supported and authored. That is, there was both object-level learning and meta-level learning in this whole classroom discussion. However, it must be noted that the learning described here pertains to the class as a whole, and not to individual students.

There were realizations authored by the students about using a basis to define a linear transformation, linear dependence and linearity of the transformation. In these narratives the students seemed fluent, that is the students were fluent in the subdiscourses. The discussion supported their meta-level learning and the adoption of the coalesced discourse.

Although the DDMT reflects the discussion, there were some other tangential discussions that do not show up on the DDMT. These discussions were about different mathematical objects than what is mapped on the DDMT. For example, at the beginning of the discussion there was a brief discussion if the set $\{a_1 = (1,2,3,4), a_2 = (0,1,0,0), a_3 = (0,0,1,0), a_4 = (0,0,0,1)\}$, suggested by a student, is linearly independent. This discussion was not mapped on the DDMT, as the narrative was from a discourse that is not apparent on the DDMT, that is the discourse of linearly independent vectors. That discourse implicitly exists underlying this DDMT and can be considered as the preceding discourse that the students were expected to have adopted. Similarly, there are narratives from within even more basic discourses, such as adding scalars or adding vectors, that are not apparent on this DDMT. The choice of which narratives to note considers what objects are the focus of the lesson and what objects are considered previous knowledge and are already objectified by the students. Thus, in this section narratives from within other, previously learned discourses were not mapped onto the DDMT.

6.2 Explorative opportunities in all the workshops

There were 11 whole class discussions recorded. DDMTs were constructed for these, in a similar method to what was described above. The whole class discussions in the various workshops included authoring realizations in varying degrees and connecting between these realizations. In this section, six DDMTs exemplifying the discussions and the characteristics that supported explorative participation made apparent by this mapping are displayed and discussed. The use of subdiscourses, how many and which, was examined as authoring realizations in multiple discourses is part of explorative participation. I also examined an aspect of the agency given to the students, as displayed by who authored the realizations and the connections, since this is also an aspect of explorative participation. Finally, explorative participation includes connecting between the realizations, and thus the links drawn on the DDMTs were examined.

6.2.1 Workshop S1, Spring 2019, Complex Numbers, Algebra 1E

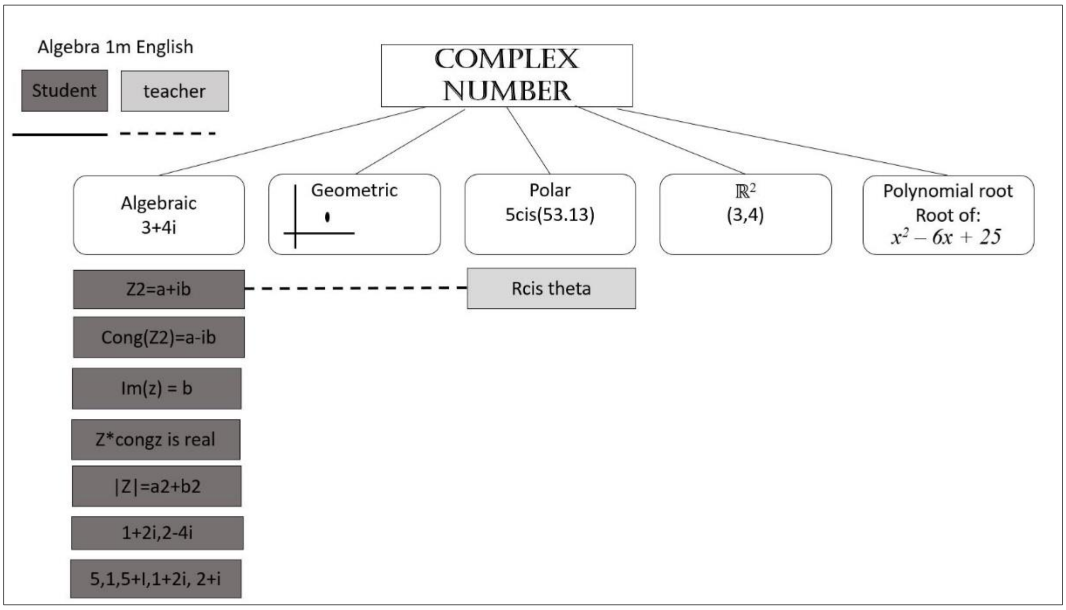

Figure 6-5 - DDMT S1 Complex numbers

Workshop S1 was in the second week of the semester and 14 students attended. The task was to prove some claims about complex numbers. The intended goal was to use the multiple types of representations and to discuss the connections between them. In general, the discussion in this workshop was focused on the metarules of logic and proof, and not on complex numbers. The first student who presented her solution to the class on the board proved the wrong direction of the claim. That is, she assumed what needed to be proved, and proved what was given. She proved z_1 = cong(z_2) => $z_1 \cdot z_2 \in \mathbb{R}$ instead of $z_1 \cdot z_2 \in \mathbb{R}$ => z_1 = cong(z_2). This led me, the instructor, to ask questions about what was given and what can be assumed. Similarly, subsequent discussions about the student’s presentations on the board focused mainly on logic and proving. This workshop was at the beginning of the semester, so the students were not yet familiar with all the meta-rules of logic and proof.

The discussion included many mathematical ideas and exposed the students to important executive meta-rules of logic and proving. However, the discussion did not include any opportunities for meta-level learning of the coalesced discourse of complex numbers, as seen by the absence of any links drawn on the DDMT. The discussion, in Figure 6-5 above, included realizations from within only a single subdiscourse - the algebraic subdiscourse. This subdiscourse is the most familiar to the students. It is included in the secondary school curriculum and does not use any trigonometric functions. As part of the conclusion of the discussion, I authored one realization in another subdiscourse and connected it to the already mentioned realizations, as seen by the single light grey realization in the polar subdiscourse.

The DDMT, Figure 6-4, shows the single subdiscourse used by the students and the single connection, authored by me, to a different subdiscourse.

6.2.2 Workshop W1, Winter 2020, Complex Numbers, Algebra A

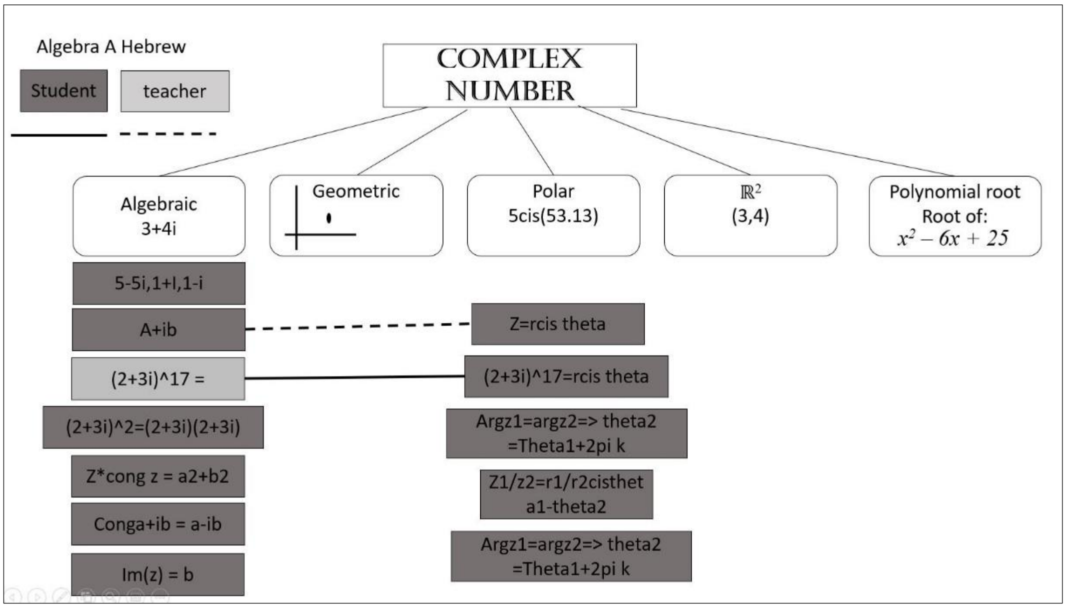

Figure 6-6 DDMT W1 Complex numbers

There were over 60 students in Workshop W1, which was in the second week of the semester. It had the same task as Workshop S1 (Figure 6-5 above) and was held in the next semester. Having become cognizant of the importance of the need to actively introduce multiple subdiscourses into the discussion and encouraging hints of them in students' talk by mapping the first workshop, I attempted to ensure that multiple subdiscourses be mentioned in this discussion. I did this in two ways. First, I asked the students how the complex number a+ib, in the algebraic subdiscourse, could be represented in another form. The students suggested the realization rcisθ in the polar subdiscourse. However, after this, the discussion reverted back to the algebraic subdiscourse, possibly since this was the most familiar one to the students.

My next attempt utilized an opportunity provided by a student's justification for a claim. As part of this justification, the statement $(2+3i)^2 = (2+3i)(2+3i)$ was authored. I attempted to ensure the discussion used the polar subdiscourse by asking how one would calculate $(2+3i)^{17}$. This question stimulated a student to give an answer using de Moivre's formula ($(rcis\theta)^n = rcis n\theta$), which can only be stated in the polar subdiscourse. Following this, more realizations in the polar subdiscourse were authored.

The discussion in W1, Figure 6-6 above, included realizations from within two subdiscourses. Yet, there were only minimal connections authored between the two subdiscourses mentioned. After the workshop, I wrote in my teaching journal, "the discussion included realizations in lots of discourses" (Journal, 5/11/2019). The DDMT showed that although this was so, there were only minimal links. That is, there were minimal narratives in the coalesced discourse. The main focus of the discussion in this workshop was also on the general mathematical meta-rules of proving and logic, as in Workshop S1, and not on the connections between the subdiscourses.

The DDMT, Figure 6-5, shows the two subdiscourses that were involved in the discussion and minimal connections between them.

6.2.3 Workshop W2, Winter 2020, Matrices, Algebra A

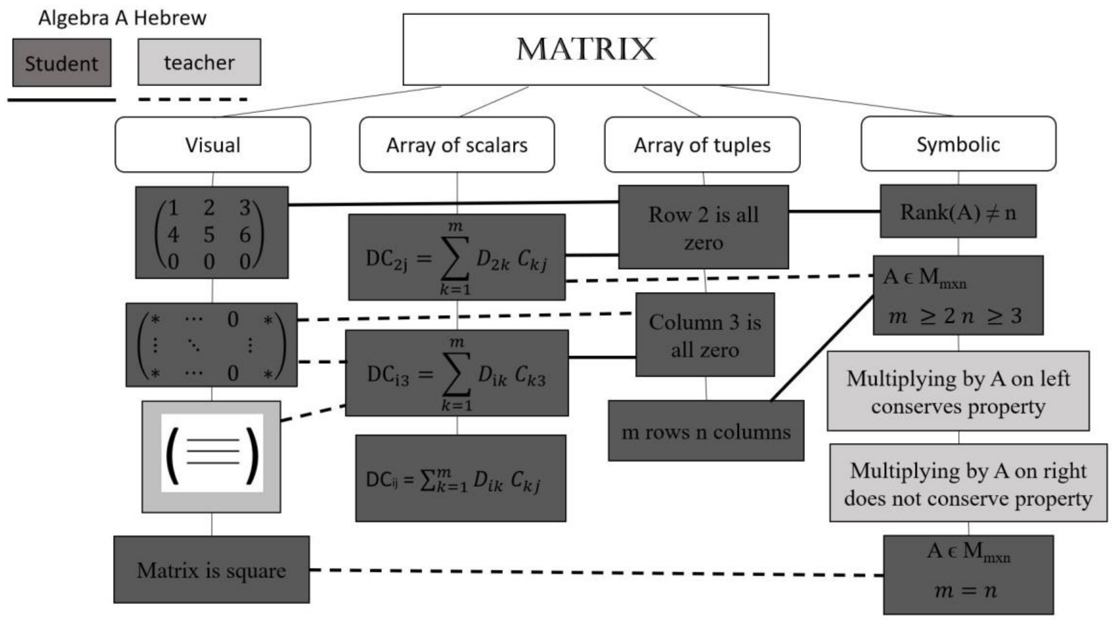

Figure 6-7 - DDMT Workshop W2 Matrices

Workshop W2 was in the fourth week of the semester and 15 students attended. The task given was to suggest a claim about a product of two matrices that had certain properties and to prove it. One matrix had a row of zeros and the other matrix had a column of zeros. The goal for the workshop was to lead the discussion to discuss the different realizations of matrices and to compare them. The discussion included many realizations and multiple links between subdiscourses.

In this workshop I ensured that multiple discourses were mentioned. This was done in several ways. First of all, I ensured that the logic of the claims and the proof was correct. At the start of each proof I asked, “what are we proving?” and “what is given?” There was minimal discussion about the metarules of logic, and so there was sufficient time for a meaningful discussion about matrices.

Another way I ensured that there were multiple subdiscourses was by actively suggesting that the students use other subdiscourses. After a student authored and proved a claim about symmetric matrices using the elements of the matrices ($DC_{ij} = \sum D_{ik}\, C_{kj}$) I asked the class, “how else can we prove it”? This elicited some mumbled suggestions, and no usable narratives in other subdiscourses were authored. I next attempted to explicitly ask for a narrative in another subdiscourse by saying, “some students used a picture with dots. Can we use that?” This justified their use of narratives in the subdiscourse of visual descriptions. Following this, a student presented a proof on the board using a picture of an array of numbers. This was followed by a discussion if this type of proof is considered acceptable, i.e.

would such a proof receive points on an exam. This discussion gave validity to other types of proofs. Following this, the students volunteered proofs of various types using narratives from multiple discourses, as seen on the DDMT (Figure 6-7).

Once the students authored multiple realizations from within multiple subdiscourses, the discussion could be focused on discussing the connections between these realizations. After multiple types of proofs were presented, I started a discussion by asking, “which type of proof is best?” This led to comparing and contrasting the different types of proofs and deliberating which type of proof is suited to what type of task. This included multiple narratives in the coalesced discourse, as comparisons between narratives in the subdiscourses were made. The multiple connections apparent on the DDMT were enabled by the existence of many realizations in multiple discourses, without which the discussion would have been pointless.

The discussion about what is considered an acceptable proof included many narratives in the new coalesced discourse. Yet, it was also about the metarules of proving and what is considered an acceptable proof. The executive metarules of proving still needed discussion, but in this case the discussion was not instead of a discussion about the topic being discussed. Thus, the discussion about the metarules of proving was harnessed to support authoring narratives in the new coalesced discourse.

The DDMT, Figure 6-7, shows multiple subdiscourses and numerous links. The realizations were mainly authored by students, and the links were authored both by the students and by me.

6.2.4 Workshop W4, Winter 2020, Linear Independence, Algebra A

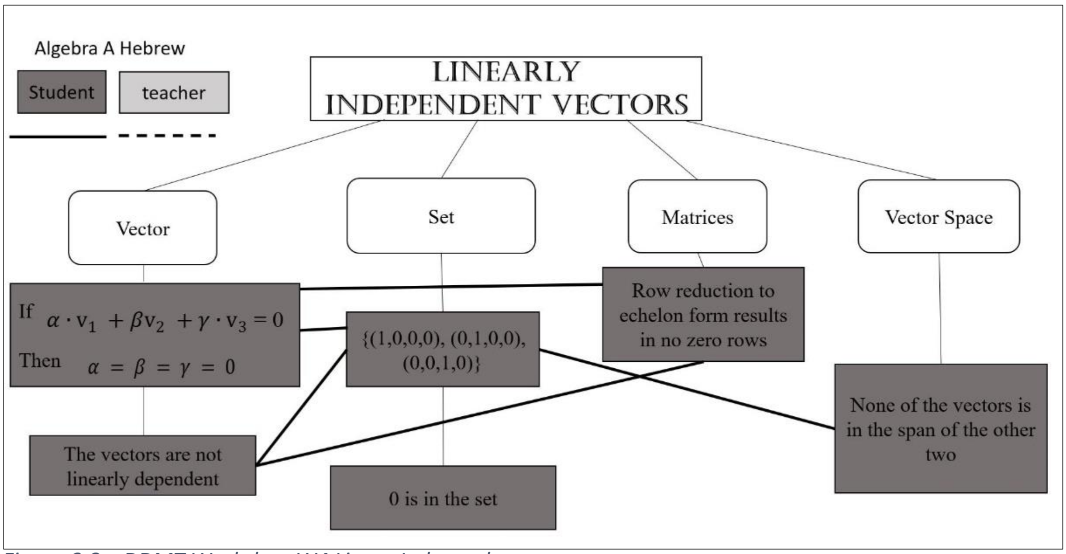

Figure 6-8 – DDMT Workshop W4 Linear Independence

Workshop W4 was held during the eighth week of the semester with 24 students on the topic of linear independence. The task asked to either prove claims or to give a counter example for these claims. The students were familiar with the narratives in the subdiscourses and were comfortable authoring narratives in the coalesced discourse.

The students' familiarity with the different realizations of linearly independent vectors could be due to the timing of the workshop. Due to technical issues, the workshop was held later than planned in the syllabus, and the students had practiced using these narratives. The students used the different discourses interchangeably and the connections between the discourses were obvious to the students. For example, a student stated "they (the vectors) are linearly independent since the matrix (whose rows are representatives of the vectors) is reduced". The first half of this narrative is from within the subdiscourse of vectors and the second half of the narrative is from within the subdiscourse of matrices. Thus, the narrative connects between two subdiscourses and is in the coalesced discourse of linearly dependent vectors. The connections were elicited from the students after I insisted on their justifying narratives by asking, "how do you know that?" and "Is that the definition of linear independence?" The students' familiarity with the various realizations allowed them to author all the connections seen on the DDMT, Figure 6-8, but they did not author these links without my asking for them.

The students seemed most familiar with the subdiscourse of matrices. They mostly justified their claims pertaining to linear dependence using narratives about echelon reduced matrices. Students stated that the matrix is reduced and did not expand their claims, as this was deemed by them sufficient proof. In contrast, statements using scalars or linear combinations to justify linear independence had continuations such as "then the matrix is reduced". That is, the students associated linear independence with realizations in the subdiscourse of matrices. This subdiscourse does not include any of the formal definitions of linear independence. It does include a very well-defined procedure for determining linear independence. The use of the matrix subdiscourse in this manner was also apparent in the other workshop held on linear dependence in the Spring semester.

To conclude, the DDMT, Figure 6-8, shows realizations in multiple subdiscourse with links all authored by the students.

6.2.5 Workshop W5, Winter 2020, Linear Transformations, Algebra A

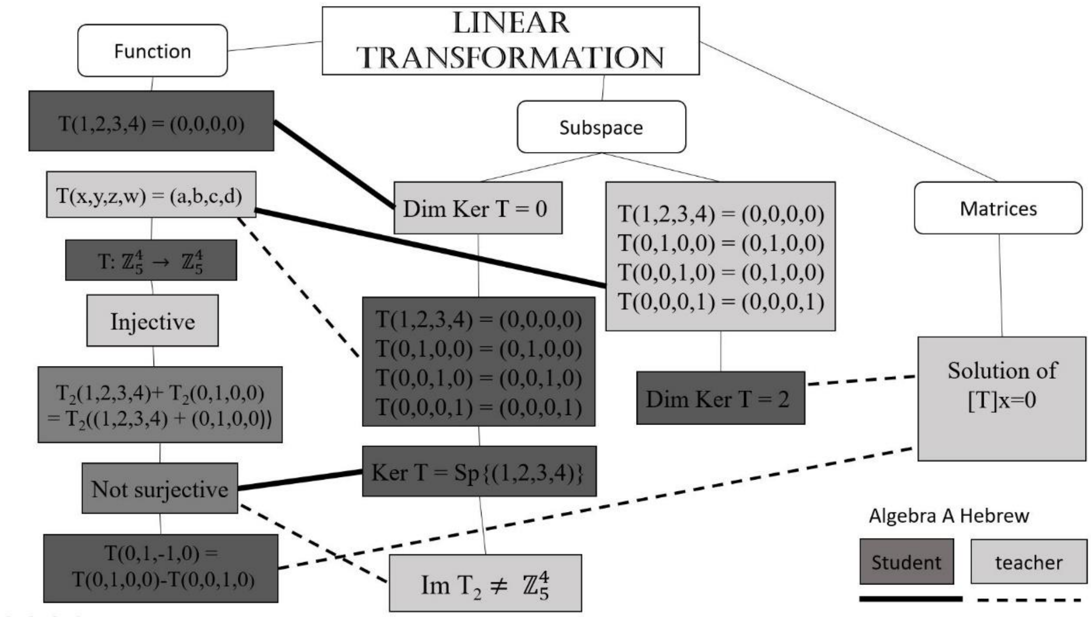

Figure 6-9 – DDMT Workshop W5 Linear transformations

Workshop W5 was in the eleventh week of the semester and 7 students attended. The construction of this DDMT was presented in the first half of this section. The discussion focused on the task of constructing linear transformations with certain properties. The main goal of this task was to demonstrate the advantages of exploring properties of linear transformations by using bases to define the transformation.

During the discussion, initially the students authored narratives from within only a single discourse, that of functions. This could be seen, for example, in the statement “it’s injective” (pertaining to the realization of the linear transformation as a function) being sufficient for the students and not needing any explanation. In contrast, for the equivalent statement “the dimension of the kernel is greater than zero” the students demanded explanation and connected it to the transformation being injective.

In the discussion, I attempted to support and encourage the students’ use of the subdiscourse of vector spaces, since it was the newest discourse for them. For example, defining linear transformations on bases (that is, in the discourse of vector spaces) is a more efficient method of exploring their properties. However, the students did this only after being prompted by me. During the small group period, I noted that the students were attempting to explore the properties of linear transformations they had constructed randomly, mainly by relying on the discourse of functions. That is, they gave the general case of the transformation, for example, T(x,y,z,w) = (x-y, x-y, x-y, x-y), and then attempted to examine if this transformation fulfilled the properties required by the question. To assist them in linking to the other discourses, I first asked leading questions, such as, “Could you define the linear transformation differently?” Sometimes, I explicitly mentioned the alternative discourse, such as when I asked, “what defines a vector space?” This sufficed for some students to then turn to using bases, yet others still did not. In those cases, I explicitly suggested moving to the discourse of vector space, by suggesting that they use a basis to define the transformation. My monitoring work in the small groups supported the use of both discourses (functions and vector spaces) in the whole class discussion, since the students had all authored narratives in both discourses while working in groups.

The DDMT also reveals the neglect of the discourse in matrices in this discussion. Although one realization in this subdiscourse was authored and the connection to the other discourses was mentioned, there was no meaningful discussion within this discourse. The matrix realization and connection to a realization in the function subdiscourse and to a realization in the subspace discourse were authored by me, and the students did not continue using any matrix realizations in their own justifications. The subdiscourse that represents linear transformations as matrices was introduced in the lectures later. It is thus possible that, although students were already quite familiar with the discourse of matrices, they were not familiar with them as realizations of linear transformations.

To conclude, the DDMT in Figure 6-9 shows that students authored narratives from within two discourses – the discourse on functions and the discourse on vector spaces.

The DDMT also shows that although I actively tried to introduce the subdiscourse of matrix representation, it was not taken up by the students.

6.2.6 Workshop W6, Winter 2020, Diagonalizable Matrices, Algebra A

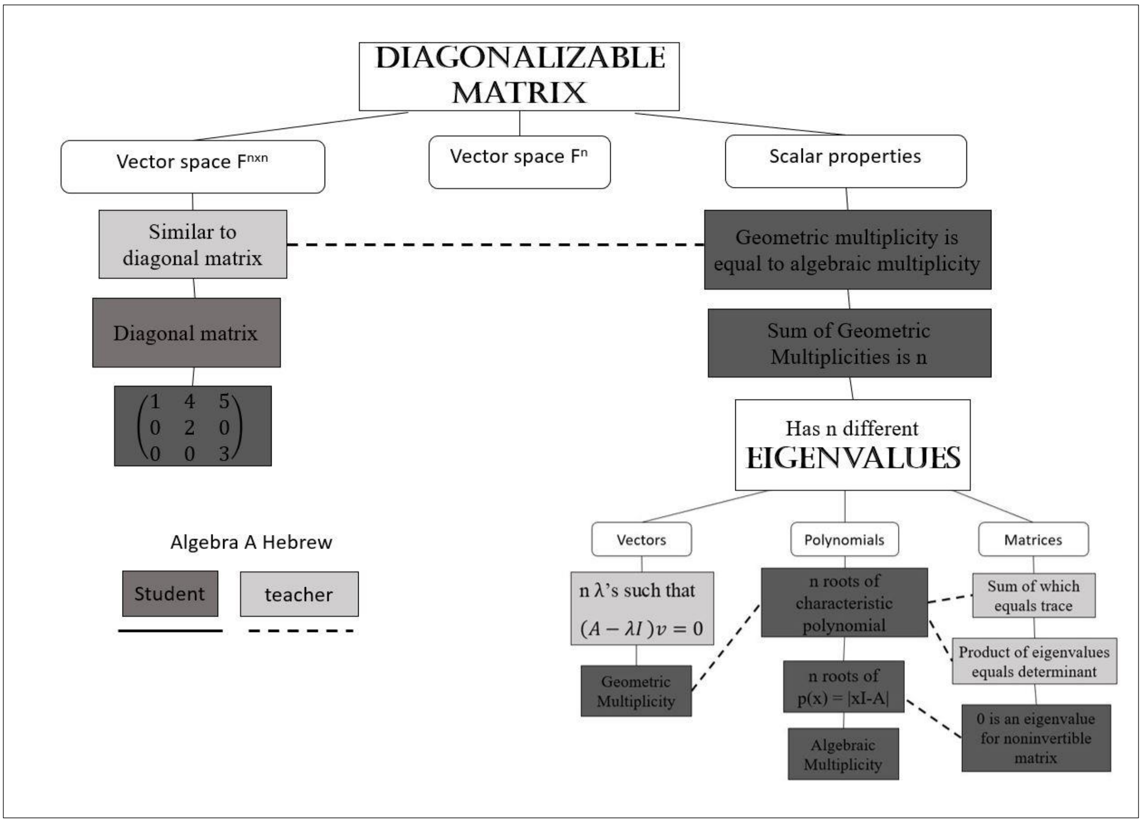

Figure 6-10 – DDMT Workshop W6 Diagonalizable matrices

Workshop W6 was in the last week of the semester and 25 students came. The task given was to give conditions on parameters such that a matrix, whose elements were these parameters, would be diagonalizable. This workshop was held during the last week of the semester, when the topics of eigenvalues and diagonalizable matrices are taught. The intended focus of the discussion was diagonalizable matrices. Therefore, the object chosen as the main node of the DMT and the DDMT was a diagonalizable matrix. Discussing this object, at that point of the course included narratives in the subdiscourses of matrices, similarity of matrices, vector spaces and scalars. For example, the narrative “there exists a basis of eigenvectors for the vector space F^n” is from within the subdiscourse of vector spaces F^n. The narrative “there exists an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$” is from within the subdiscourse of the vector spaces F^{nxn}. These narratives use eigenvalues and eigenvectors, but as an underlying discourse, as discussed when the DMT was presented in Section 5.3.6.

At the time of the workshop, most of the students had only been in lectures, and not in tutorials, about eigenvalues and diagonalizable matrices. None of them had done homework problems on either of these. The students were not sufficiently comfortable with the narratives about eigenvalues, and they needed support to author narratives within the subdiscourse of eigenvalues. The discussion was not on the intended object – diagonalizable matrices. It was about eigenvalues, an object which is part of a subsumed discourse, and is mapped on a subtree of the main tree. The discourse of diagonalizable matrices includes object-level narratives that are meta-level narratives in the eigenvalue discourse. For example, the narrative *all the roots of the characteristic polynomial are of multiplicity 1, therefore the matrix has eigenvalues with algebraic multiplicity 1*. This narrative is in the coalesced discourse of eigenvalues, as it is constructed from a narrative in the subdiscourse of polynomials and from a narrative from within the subdiscourse of matrices. This same narrative is an object-level narrative within the subdiscourse of the scalar properties of diagonalizable matrices. That is, the discussion focused on object-level narratives from within a subdiscourse, instead of on the meta-level narratives from the coalesced discourse of diagonalizable matrices. The scalar properties subdiscourse includes procedures that are familiar to the students, such as finding roots of a polynomial. This could be the reason the discussion was focused on that subdiscourse, and not on the other two available subdiscourses for diagonalizable matrices.

The DDMT, Figure 6-9, shows that the discussion included realizations from within two subdiscourses on the main tree – the subdiscourse of vector spaces F^{nxn} and the subdiscourse of scalars and polynomials. In addition, three subdiscourses were used on the subtree of eigenvalues – the subdiscourse of vectors, the subdiscourse of scalars and polynomials and the subdiscourse of matrices. In the main tree there is only one connection, authored by me. In the subtree of eigenvalues there are multiple connections also authored by me. The students were not familiar with the narratives from within the subdiscourses, and the small number of links authored in the discussion were authored by me.

The DDMT for the other workshop on diagonalizable matrices, which was given in the spring semester, Workshop S5, showed a similar picture. That is, most of the realizations and connections were in the subtree about eigenvalues, and not about diagonalizable matrices, and the links were mostly authored by me.

The DDMT showed that when the students were not sufficiently familiar with the underlying mathematical objects, the discussion turned to the underlying objects. The discussion provided opportunities for narratives in a coalesced discourse, however the discourse was not the expected one.

6.3 Summary of the DDMT analysis

The 6 DDMTs shown above exemplify the DDMTs of all 11 workshops. In most cases, the implementation of the tasks included numerous opportunities for authoring realizations in multiple discourses and for authoring connections between these realizations. That is, there were opportunities for the students to participate exploratively. Additionally, there were opportunities for the students to author narratives in the new, coalesced discourse which supported meta-level learning. I now turn to some general observations about the workshops that resulted from the DDMT analysis.

In most of the DDMTs there is a single branch that is more densely filled than the others or most links lead to a single branch. This shows that from within the discourses available for each topic, there was usually one discourse that was more dominant in the discussion. Thus, students explained their ideas and justified their claims by linking to narratives in this discourse and relied on this discourse as the base of their narratives. In some cases, this dominant discourse was more familiar to the students. For example, in the linear transformations workshop, Workshop W5 (Figure 6-8), the students justified claims with narratives from within the functions discourse, probably since it was familiar to them from secondary school and from their calculus courses. In other cases, the dominant discourse had clear procedures. For example, in the workshop on linear dependence, Workshop W4 (Figure 6-7), the students' justifications about linear dependence were mostly in the discourse of matrices, where reducing to echelon form was a well-rehearsed procedure familiar to the students from lectures, tutorials and their homework. The students clung to the dominant subdiscourse and to familiar procedure within it, and the use of other subdiscourses had to be encouraged actively.

In the discussions that had more student authored links there were also more realizations authored by the students. In those discussions, the students were familiar with the narratives within the discourse, and thus the discussion could be focused more on linking between the discourses. For example, in the workshop on matrices, Workshop W2 (Figure 6-6) once the students authored multiple realizations from within multiple discourses, the discussion could be focused on discussing the connections between these realizations. Conversely, in the workshop about diagonalizable matrices, Workshop W6 (Figure 6-9), where the students were not familiar with an object from within the discourses, the small number of links authored in the discussion were authored by me. The discussion in that case was focused on the narratives from within the subdiscourses. Thus, construction of links between subdiscourses were dependent on the familiarity of the students with the narratives within the subdiscourses.

The DDMT analysis also highlighted how the instructor provided opportunities for explorative participation during the discussion. First, the focus of the discussion was guided by the instructor's questions. When the students were not familiar with general mathematical meta-level rules, I changed the focus of the discussion to the missing meta-rules. This was seen, for example, in the workshop about complex numbers, Workshop S1 (Figure 6-4), where the students were not familiar with the metarules of logic and proof. In those discussions, the focus was on these metarules and not on linking between the different discourses of complex numbers. Similarly, when the students were not familiar with the object-level narratives of a discourse, the discussion was focused on the narratives and connections within the discourse. This was the case in the workshops about diagonalizable matrices, Workshop W6 (Figure 6-9), where the discussion was mostly focused on the underlying discourses of eigenvalues and not on the discourse of diagonalizable matrices. In discourses that students were sufficiently familiar with, focusing the discussion afforded the students opportunities for explorative participation in the new discourse.

The instructor also provided opportunities for explorative participation by ensuring that the discussion included multiple discourses. In some of the workshops the students authored narratives from within only a single discourse. Other discourses were introduced by me through authoring narratives from within these discourses and by implicitly introducing other discourses through questions. For example, initially in the workshop about matrices, Workshop W2 (Figure 6-6), the students authored narratives from within the discourse of scalars and the general element of matrices. I asked the class questions in an attempt to elicit realizations in other discourses and I also explicitly requested a realization in the visual array of numbers discourse. Thus, this workshop included multiple discourses, and the opportunity for explorative participation was afforded to the students. The multiple subdiscourses included in the discussion also afforded opportunities for linking between these subdiscourses, which is an aspect of explorative participation and supports meta-level learning.

When examining the DDMTs it becomes apparent that, mostly, many of the links between discourses were authored by me, the instructor. In places where students authored links on their own, these were usually elicited by leading questions from me, asking for justifications such as “how do you know?” and “is it always true?” Thus, the instructor’s contributions and prompts can be crucial in connecting between multiple discourses, which is an integral part of explorative participation and meta-level learning.

7 Learning processes involved in a dyadic mathematical discussion without the support of an expert

The linear algebra workshops are constructed of mathematical tasks, instructor actions and student involvement. In the first section, I examined what potential the tasks afford for learning linear algebra. In the next section, I analyzed the whole class discussions and examined to what extent opportunities for meta-level learning were taken up. I examined in what ways meta-level learning was supported by the workshops, and how the instructor encouraged this. Yet the main part of students' independent exploration and struggle with the tasks took place in the collaborative learning phase of the workshops. Additionally, the recorded discussions between the students in the small groups could better expose their individual discourse than the analysis of multiple participants' discussions. Thus, I analyzed the discourse involved in the students' collaborative interactions in the small group discussions to examine the learning processes involved in a collaborative learning episode without the support of an expert.

This section utilizes a commognitive discourse analysis of dyadic interactions to examine collaborative learning in small groups and the processes involved. This chapter first presents an overview of the workshop and then examines the learning processes of a seemingly successful, egalitarian interaction by examining how a pair of students' routines changed during such an interaction. The pattern of interpersonal communication is next explored, as ineffective communication can hinder collaborative learning. Next, the objects and object related metarules involved in the discussion are analyzed to examine how these shaped the discussion. Finally, the interaction between a second pair of students is presented.

7.1 The linear dependence workshop and the task situation

The discussion between the students was in the context of a workshop, the details of which, obviously, impacted the discussion. These are presented in this section as a background and as the context of the students' discussions. The two episodes are from different semesters, yet they both are from workshops that dealt with the topic of linear dependence. All the workshops began with a reminder of the basic definitions and theorems that were presented in lectures and tutorials. The basic definition of linear dependence was written on the board in the appropriate language, and it was on the board during the students' discussions that are analyzed in this section.

The definition written on the board:

> V is a vector space over F. The set $\{v_1,...,v_n\} \subset V$ is **linearly dependent** over F if there exist $\alpha_1,\dots\alpha_n \in F$, not all zero, such that $\Sigma\alpha_i v_i = 0$. Otherwise, the set is **linearly independent**.

After the introduction, the students were presented a task printed on a paper to solve in small groups of two or three students. The task given to the students included four assertations to determine if they were true or false. In the Spring semester the instructions on the worksheet, which was in English, stated:

> V is a vector space over the field F. Are the following statements true or false?
>
> If a statement is true, prove it.
>
> If a statement is false, give a numerical counter example.

In the following semester, based on what occurred in the Spring semester, the worksheet was modified, and the following line was added to the instructions:

> If a statement is sometimes true, give an example for which it holds and an example for which it does not hold.

The four assertations, in both semesters, were:

1. $\{u_1, u_2, u_3\} \subseteq V$ is a linearly independent set and $u_4 \in V$, then the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent.
2. $\{u_1, u_2, u_3\} \subseteq V$ is a linearly dependent set and $u_4 \in V$, then the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent.
3. $\{u_1, u_2, \ldots, u_6\} \subseteq V$ is a linearly dependent set, then $\text{Span}\{u_1, u_2, \ldots, u_5\} = \text{Span}\{u_2, u_3, \ldots, u_6\}$.
4. $\{u_1, u_2, \ldots, u_6\} \subseteq V$ is a linearly independent set, then $\text{Span}\{u_1, u_2, \ldots, u_5\} = \text{Span}\{u_2, u_3, \ldots, u_6\}$.

The analysis presented in the next sections examines the students' discussion of the proofs of the assertations. The two pairs of students, from different semesters, worked on the same task. Assertation 2, from the above task, always holds and can be proved in many ways. It can be proved succinctly by using the theorem that states that a set including a linearly dependent set is linearly dependent. Thus, the set $\{u_1, u_2, u_3, u_4\}$, which includes the linearly dependent set $\{u_1, u_2, u_3\}$, is linearly dependent. This proof, although efficient, does not give any intuition and requires familiarity with that specific theorem. A more detailed proof using only the basic definition of linear dependence starts with the given that $\{u_1, u_2, u_3\}$ is a linearly dependent set. Thus, according to the definition of linear dependence, there exist scalars, $\alpha, \beta, \gamma \epsilon \mathbb{R}$, not all zero, such that $\alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3 = \vec{0}$. Since $0 \cdot u_4 = \vec{0}$, it also holds that $\alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3 + 0 \cdot u_4 = \vec{0}$. That is, there exists a linear combination of the four vectors whose scalars are not all zero, and the set, $\{u_1, u_2, u_3, u_4\}$, is linearly dependent.

7.2 Hadar and Yaniv – an egalitarian pair with a seemingly successful collaborative learning session

Hadar and Yaniv (pseudonyms) were students in the Algebra A course in the Winter 2020 semester. They chose to participate in almost all the offered workshops. They were both first semester students studying towards a degree in computer science. They were sitting near each other when they were asked to work in small groups, and so they worked together. They did not have any prior acquaintance with each other.

As described more fully in the methods section, a preliminary analysis of all the recordings yielded Hadar and Yaniv's interaction as potentially illuminating. The two students both authored mathematical narratives, they both questioned the other's claims and they both seemingly advanced in some aspects of solving the task. Moreover, the initial observations revealed that Hadar and Yaniv explicitly disagreed at the beginning of their interaction and later advanced to a collaboratively constructed narrative, which I assessed as canonical. Thus, the pair's narratives seemingly advanced through learning during their interaction. The processes of this seemingly productive, joint interaction could shed light on learning processes in a successful, collaborative interaction. Therefore, the dyadic interaction between Hadar and Yaniv was analyzed in depth.

7.3 The dyadic mathematical discussion

Altogether, Hadar and Yaniv worked in a pair for 10 minutes. First, they quickly solved the first task by giving an example for which the first assertation does not hold and an example for which it does hold.

1) True or false: $\{u_1, u_2, u_3\} \subseteq V$ is a linearly independent set and $u_4 \in V$, then the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent.

Hadar suggested they use $\mathbb{R}^4$ and the linearly independent set {(1,0,0,0), (0,1,0,0), (0,0,1,0)} as an example of the set with three elements. Then, the pair discussed if an "abstract answer" would be acceptable or do they need to use actual numbers. Hadar decided they need to use actual numbers and suggested a vector, "*we take zero in the last place then… and something else and then linear independence*". The example she gave for a linearly independent set, when the assertation holds, was {(1,0,0,0), (0,1,0,0), (0,0,1,0), (5,0,0,2)}. The pair did not explicitly state an example for when the assertation does not hold, but it was clear they considered it as possible. For this possibility Hadar described a vector for which the "last place" has a zero. This implied that such a set is a simple and obvious example of an instance where a vector added to the first set yields a set with four vectors, that is a linearly dependent set. For example, the set {(1,0,0,0), (0,1,0,0), (0,0,1,0), (5,0,0,0)}.

The pair next turned to the second assertation, on which most of the rest of the interaction was spent. There they disagreed. The assertation was:

2) True or false: $\{u_1, u_2, u_3\} \subseteq V$ is a linearly dependent set and $u_4 \in V$, then the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent.

7.3.1 Task situation, initial routines and co-constructed final chain of narratives

The full transcript of the pair's discussion of this assertion is in Appendix C1, Section 10.3.1. This section brings excerpts of the transcript when the exact formulation and words are important for the analysis, otherwise a synopsis of what happened is provided. The pair's discussion started with the following excerpt.

27	Yaniv:	Yes. It (*the assertation*) is definitely true.
28	Hadar:	A linearly dependent set, u belongs to V, all these together ($\{u_1, u_2, u_3, u_4\}$) are linearly dependent…Are you sure it's (*the assertation*) true?
29	Yaniv:	If it ($\{u_1, u_2, u_3\}$) is already linearly dependent, and we add another vector, this subset ($\{u_1, u_2, u_3\}$) is still linearly dependent.

From the beginning of their work on this assertation, the students had different tasks. Yaniv's task was to prove that the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent. He claimed that the assertation is true [27], that is the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent and he gave a justification for his claim [29]. This justification hinted at the procedure he used to determine if a set is linearly dependent. This procedure used a theorem proved in the lecture - that a set including a linearly dependent subset is a linearly dependent set. Thus, Yaniv's initial routine was using the procedure of finding a linear dependent subset to solve the task of proving that the set is linearly dependent.

In contrast to Yaniv, Hadar's initial task was to show the assertation was false. She attempted to do so by proving that there exists a vector u_4 such that the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent when the set {u_1, u_2, u_3} is linearly dependent. To do this, Hadar suggested the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)} as a counter example to the assertation. To justify her claim, Hadar used an idiosyncratic procedure, the details of which will be described later in this section, where she explored the status of each vector in the set, determining whether it was "linearly dependent" or not. Hadar's initial routine was to use her idiosyncratic procedure to solve the task of suggesting a counter example.

Hadar and Yaniv's final co-constructed proof of the assertation differed from each of the individually authored proofs. Moreover, the final chain of narratives included pieces of both initial narratives and new narratives that they both agreed to. The process of co-constructing a proof started with Hadar claiming that there was a way to construct a counter example to the assertation:

38 Hadar: The 3 (vectors) are (linearly dependent). But the fourth isn't. So, the entire set is linearly independent

39 Yaniv: Why?

40 Hadar: Because…Because it's possible (to construct such a set). You can bring u1=(1,0,0,0) ; u2=(2,0,0,0); u3=(3,0,0,0); u4=(0,1,0,0) [writing this example as she talked]

41 Yaniv: Then it ({u_1,u_2,u_3,u_4}) is still linearly dependent.

42 Hadar: How is it linearly dependent?!

43 Yaniv: No, it (the vector) isn't – but the set altogether is.

44 Hadar: Why? If you find scalars, that not all of them are zero…?

45 Yaniv: That means that it is linearly dependent

46 Hadar: And this (the linear combination) won't be equal to zero, because this (u4), you cannot neutralize if you don't put a zero for him

47 Yaniv: Yes. But it doesn't matter if it (the scalar multiplying u_4) will be zero, if all the rest uh…if there is one

This excerpt displays Hadar's idiosyncratic procedure for determining if a set is linearly independent. In [38] she hints that, for her, linear independence is a property of single vectors ("*the fourth isn't (linearly dependent)*"). Hadar's procedure involves examining the status of each vector in the set, and if at least one vector is linearly independent then she concludes that the set is linearly independent. Her procedure for examining the "linear dependence" of each vector is further revealed in her statement, "*this (the vector (0,1,0,0)), you cannot cancel out if you don't put a zero for it*" [46]. This procedure examined whether the scalar used to "cancel out" the vector (0,1,0,0) is 0, and if so, determined that the vector (0,1,0,0) is a "linearly independent" vector.

In response to Hadar's suggestions, Yaniv continued to insist that the set she was suggesting was still linearly dependent [47]. However, he did not object to Hadar's procedure of "canceling out" vectors by checking which scalars "cancel them out". Following some more discussion, where Yaniv convinced Hadar that her procedure could still lead to linearly dependent sets, Hadar backed up from her original suggestion and instead used her procedure of "canceling out" vectors to prove Yaniv's claims.

78 Hadar: Then…if we have all sorts in the set, and we put for all of them zero (we put scalars that would nullify them), but for the zero vector we put 3…

79 Yaniv: That's it. Exactly.

80 Hadar: Then the set becomes?

81 Yaniv: Linearly dependent.

82 Hadar: Dependent.

83 Yaniv: Yes.

84 Hadar: OK. That's the idea. The idea…exactly the conclusion at the end.

After this discussion, Hadar changed her narrative to align with Yaniv's (canonical) narrative. She said, "Then wait a second…then…Wow! It's hard to realize this. That it ({(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)}) will always be dependent…We are saying it's (assertation 2) always true" [86]. Yaniv agreed and suggested that they formalize their proof, "It's true. We need to prove it" [87]. Now, their tasks of formalizing the proof that the assertion is always true aligned and they concentrated on constructing this formal proof. Hadar seemed to be a full participant in the authoring of this formal proof. She started by stating, "We can do…if we said all these have an alpha 1, alpha 2" [94] and writing on the worksheet. She then clarified this and used terminology conforming to the language used in formal definitions – "alpha1" and "that is".

100 Hadar: That is, there exist scalars such that the sum of this set ({(1,0,0,0), (2,0,0,0), (3,0,0,0)}) will be equal to zero, even if they (all the scalars) are not equal to zero.

101 Yaniv: Exactly.

102 Hadar: And then if we add another vector, we can multiply it (the additional vector) by zero (and the sum of all vectors will still remain zero).

7.3.2 The changes in the pair's routines during this interaction

Yaniv's initial routine was canonical and would probably have been deemed acceptable and sufficient by an expert mathematician. After all, his task aligned with the canonical solution and he used an appropriate procedure from a theorem in an appropriate place. However, when looked at more closely, Yaniv's narratives around this routine were relatively thin. He did not justify why this was an appropriate theorem or why it is true, he just repeated the theorem as a justification. In answer to Hadar's question, "*are you sure*?" [28], he answered, "*we add another vector, this subset is still linearly dependent*" [29]. Hadar tried to convince Yaniv that there can be a counter example and he refuted her claim by restating the theorem, "*If we add, doesn't matter what we add…these 3 vectors will still be dependent*" [37]. This was also his answer to Hadar's counterexample. He said, "*Then it is still linearly dependent*" [41], without any details or justifications. Thus, Yaniv did not connect the theorem he used to the definition of linear dependence displayed on the board, nor did he suggest any other methods of solving this task in response to Hadar's questions. Hadar continued to question Yaniv, "*How is it linearly dependent*?!" [42]. She also suggested using the definition, "*Why? If you find scalars, that not all of them are zero*…?" [44]. Following this, Yaniv attempted to connect his narratives to the definition, which uses scalars, "*But it doesn't matter if it will be zero, if all the rest* …" [47]. Hadar's questioning of Yaniv's initial individual routine compelled him to bond his canonical routine to the definition familiar to them both by clarifying and elaborating his original narratives.

Concerning Hadar, her original routine was non-canonical. During the interaction with Yaniv, she changed her task to align with Yaniv's canonical task. Additionally, she modified her procedure. Her original procedure included cancelling out vectors by multiplying by zero. Her modified procedure acknowledged the possibility of non-zero scalars. She also reported comprehension after this interaction, "*OK. That's the idea*." [84] and expressed awareness of the change she made to her own narratives "*Wow! It's hard to realize this*." [86].

7.3.3 Did Hadar's routines really change? Examining her performance on the next task

In the above-described transcript, Hadar ultimately agreed that the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)} is a linearly dependent set. This may indicate that she learned something new, in the sense that her routines and narratives changed to align more with canonical ones. However, in the pair's discussion of the next assertion, it became evident that the situation was more complex. There, Hadar still used her previous routine, which explored the status of each vector separately. Not only that, but she also reverted to stating that the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)} is a linearly independent set.

The context of this surprising turn of events was the pair's attempts to prove the next assertion:

3) $\{u_1, u_2, \ldots, u_6\} \subseteq V$ is a linearly dependent set, then Span$\{u_1, u_2, \ldots, u_5\}$ = Span$\{u_2, u_3, \ldots, u_6\}$.

Hadar began their discussion by suggesting that this was true, since "*u_1 can be expressed as a combination of u_2 through u_6, so we can take no notice of it*". Yaniv questioned this by asking if maybe u_1 was not "*the vector that can be ignored*". Hadar answered, "*Does there exist such a vector? If the set is linearly dependent ... can't each vector be expressed as a linear combination of the others*?" Again, we see Hadar examining properties of single vectors (being expressed as a linear combination of others) when the property given in the assertion pertained to the set $\{u_1, u_2, \ldots, u_6\}$.

After some discussion about this between Hadar and Yaniv I came to monitor their group. Hadar asked me, "*We are wondering about how a vector can be left out of a linearly dependent set. Our question is - is it any of the vectors*?" I answered, "*not necessarily the first or the last*," and then suggested they try to construct a concrete example of such a set and explore this. After I left, Yaniv suggested they use the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)}, which they had discussed for the previous assertion, and pointed to it written on their worksheet. Hadar's reaction to this suggestion is in the following excerpt.

220	Hadar:	No. but that (the set (1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)}) is not dependent.
221	Yaniv:	Yes. It is a dependent set.
222	Hadar:	Only the first three (vectors) are (linearly dependent).
223	Yaniv:	No, all together they are dependent.

Hadar's claim in [220] is quite surprising, given that she had just authored the opposite narrative during the pair's discussion of assertion 2. Additionally, her justification of this claim in [222] still uses her initial idiosyncratic procedure, authored before the discussion with Yaniv. The above lines show that, at least for Hadar, the discussion in pairs was not productive for changing her idiosyncratic ways of treating linear dependence.

The collaborative learning episode did not advance Hadar's learning. Studies have suggested that social interactions and ineffectual communication can hinder collaborative learning (Ben-Zvi & Sfard, 2007; Sfard & Kieran, 2001). Thus, next I turn to examine the possibility that Hadar's lack of advancement was due to ineffectual communication, as suggested by these studies.

7.4 Pattern of Interpersonal Communication

To understand whether the problems in advancing Hadar's discourse were due to ineffective interpersonal communication, I examined the pair's mutual engagement through their use of the different channels of communication available. These were analyzed by first segmenting the transcript of their discussion about the task into mathematical narratives. This segmentation allowed the examination of how each pair listened to each other's mathematical ideas - if they were attending to the mathematical content of each other's narratives and to what extent they were contributing to the discussion.

The channels of communication between Hadar and Yaniv were labelled and colored according to the following table.

Private
Interpersonal Reactive
Interpersonal Proactive

In Section 7.3.1, above, I showed an excerpt where Hadar suggested that a linearly independent set can have a linearly dependent subset. Yaniv did not agree that this is possible, and Hadar attempted to explain and justify her claim. I now exemplify the interpersonal communication between Hadar and Yaniv on this same segment.

	Speaker	Verbal	NonVerbal	Hadar's Channel	Yaniv's Channel
39	Hadar	It's possible (that a linearly independent set would have a linearly dependent subset)		Reactive Interpersonal	
40		We can bring (*as an example of a linearly independent set*) u_1 as (1,0,0,0), and u_2 as (2,0,0,0), u_3 as (3,0,0,0) and u_4 as (0,1,0,0)	Writing example on paper	Proactive Interpersonal	

41	Yaniv	Then it (the set of 4 vectors) is still linearly dependent	Pointing to paper		Reactive Interpersonal
42	Hadar	How can it ((0,1,0,0)) be linearly dependent?		Reactive Interpersonal	
43	Yaniv	No, not it ((0,1,0,0)) by itself,			Reactive Interpersonal
44	Yaniv	But the entire set together	Looking at Hadar		Proactive Interpersonal
45	Hadar	Why?		Reactive Interpersonal	
46	Hadar	If you find scalars, that not all of them are zero	Pointing to paper	Proactive Interpersonal	
47	Yaniv	Yes.			Reactive Interpersonal
48	Yaniv	But it doesn't matter if he will be zero, if all the rest uh…if there is one …			Proactive Interpersonal

Table 7-1 Classification of Hadar's and Yaniv's channels of communication

Hadar's reactions to Yaniv's statements ([39], [42], [45]) were classified as using the interpersonal reactive channel, since these utterances were a reaction to Yaniv's reasoning and ideas. Hadar also authored narratives of her own and asked for Yaniv's input about these narratives in the proactive interpersonal channel ([40], [46]), where her utterances were aimed at getting a reaction from Yaniv. He responded to Hadar's questions in the reactive interpersonal channel ([41], [43], [47]) by relating directly to the mathematical content of Hadar's narrative. Yaniv then responded in the reactive interpersonal channel and asked for a response to another claim in the proactive interpersonal channel ([44], [48]).

This analysis shows that both Hadar and Yaniv attended to each other's mathematical narratives and justified their disagreement by relating to the content of the other's claims. Hadar did not follow Yaniv's claims blindly, rather she authored independent narratives of her own to which Yaniv listened and the pair discussed. For example, Hadar suggested in [40] a set of vectors and the pair discussed this set's properties. This pattern of communication was repeated throughout the interaction. The coding of their entire interaction, in Appendix C-2, Section 10.3.2, displays that most of the pair's discussion was in the interpersonal channel. Additionally, almost each narrative of Hadar and Yaniv was split into a reactive narrative followed by a proactive narrative. That is, first each of the pair responded to the other's utterance and then stated something new, asking for a reaction.

To conclude, this analysis showed that the initial impression of egalitarian interaction was justified. Moreover, it shows that the interaction was full of proactive and reactive communications on the part of both of the parties. There was minimal use of the private channel in Hadar and Yaniv's discussion, thus they had relatively full access to each other's routines for solving the task, they were both engaged in the discussion, and they both considered the others' ideas as worthy of consideration. Thus, the communication supported collaborative learning and it is difficult to blame Hadar's ineffective learning on any interactional features of their discussion.

7.5 Hadar and Yaniv's objectification processes

In this section I turn my analytical gaze to the more tacit rules, or metarules of the mathematics underlying this task, to examine whether those could explain the ineffectiveness of Hadar's learning. In Chapter 5, I showed that embedded in the task Hadar and Yaniv were working on is the mathematical object "set of vectors". This object can be realized in four subdiscourses – vectors, sets, matrices and vector spaces. The analysis in Chapter 5 pointed to the fact that solving this task necessitates using at least two of the subdiscourses - vectors and sets - and connecting between them. Therefore, to understand the roots of the persistence of Hadar's idiosyncratic and non-canonical narratives I next turn to the subdiscourses Hadar and Yaniv used in their solution process. I do so by mapping their discourse on DDMTs, similar to those constructed for implementations of this task in Chapter 6.

The DMT constructed in Chapter 5, on which the DDMT in Chapter 6 was based, was for the object "set of linearly independent vectors". A set of linearly *dependent* vectors is a set which is *not* linearly independent, and thus uses the same subdiscourses, as was explained in Chapter 5. The students' narratives were mapped onto a DDMT, as was done for the whole class discussion in Chapter 6, where the method is described in detail. Yaniv's narrative pertained to a set of linearly dependent vectors, and so this was used as the node of the DDMT mapping his narratives. Hadar's narratives pertained to both linearly dependent vectors and to linearly independent vectors, and so two separate DDMTs were constructed for her narratives.

7.5.1 Yaniv's DDMT

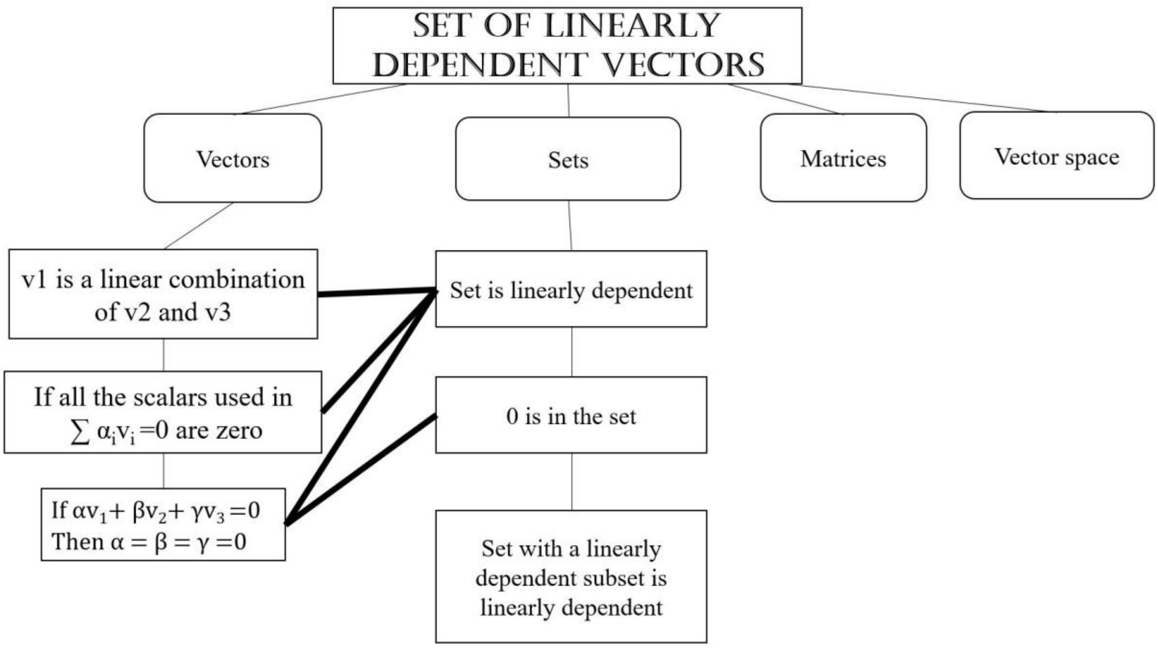

Figure 7-1 Yaniv's DDMT for set of linearly dependent vectors

As can be seen in Figure 7-1, Yaniv authored realizations in the subdiscourse of vectors and in the subdiscourse of sets. He also authored links connecting between these subdiscourses. For example, Yaniv answered a suggestion of Hadar's saying, "*But it doesn't matter (for the set's linear dependence) if it (the scalar multiplying u_4) will be zero, if all the rest (aren't)*" [47]. By this statement, he connected between a narrative in the subdiscourse of vectors about the scalars multiplying the vectors in a linear combination with a narrative in the subdiscourse of sets about the set's linear dependence. He also said, "*If there is one scalar at least that is different from zero...then it (the set) is (linearly dependent).*" [53]. This narrative also includes a narrative from the vectors subdiscourse, "there is one scalar at least", and a narrative from the set subdiscourse "it (the set) is (linearly dependent)". These narratives, which connect between the subdiscourses, are from within the coalesced discourse of sets of vectors. Thus, the objects embedded in Yaniv's narratives are from within this discourse, that is, the objects Yaniv discussed were "sets of vectors".

The DDMT showed that Yaniv's narratives pertain to the mathematical object of a set of vectors. This is also noticeable in the narratives. When Yaniv discussed the property of linear dependence of a set he consistently used the singular pronoun *it* to refer to this object, for example, "*If **it** is already linearly dependent*" [29] and "*that means that **it** is (linearly dependent)*" [45]. This suggests that he had encapsulated the different vectors into a single set, and he treated the set of vectors as an object, rather than as only a collection of objects (vectors). His narratives also referred to sets of vectors as an object with properties, and not as the result of a procedure. For example, "*the set altogether is*" [43] and "*if zero is in the set*" [70]. Yaniv's routine, to examine the set for specific subsets, also pertained to the object "*a set of vectors*".

7.5.2 Hadar's DDMTs

Hadar authored non-canonical narratives and some of her narratives were authored with Yaniv's support and as a result of her interaction with Yaniv. This is noted in the DDMT by filling in the boxes of these types of narratives in different colors. As explained above, there are two DDMTs for Hadar – one mapping her narratives pertaining to linear dependence and one mapping her narratives pertaining to linear independence.

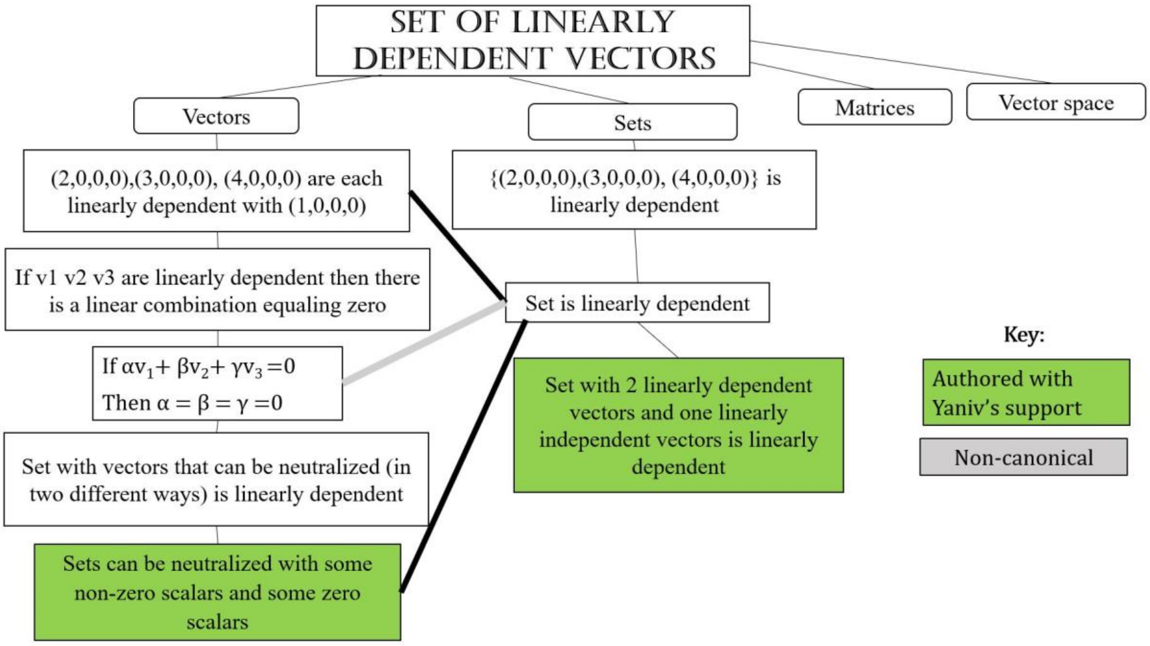

Figure 7-2 Hadar's DDMT for set of linearly dependent vectors

The DDMT of Hadar's narratives about linearly dependent sets shows that these are mainly in the subdiscourse of vectors. She authored by herself two narratives in the sets discourse, including one narrative not connected to any justification, "the set is linearly dependent". The other narrative she authored was her "prototype" of linear dependent sets. Hadar said, "*3 like this (u₁ = (1,0,0,0))*" [30] and explained "*Three that are dependent with u_1. Let's say here (the first component of the vector) is 2,3 and 4.*" [36]. Thus, Hadar's narrative was that the set {(2,0,0,0), (3,0,0,0), (4,0,0,0)} was linearly dependent.

Two of the narratives that appear in the DDMT were authored with Yaniv's support. This was shown above in Section 7.3. With this support, Hadar connected between a narrative in the vector subdiscourse and a narrative in the set subdiscourse. However, the narrative in the vector subdiscourse was a small modification of her idiosyncratic procedure and did not herald any major change in her routine. She also authored a non-canonical link between the subdiscourses, shown in the figure above as a grey line. She said, *"(linear dependence) means that alpha1 is equal to alpha 2 is equal to zero, they are all equal to each other and they are equal to zero*" [72]. This is part of the definition of linear **in**dependence, and not linear dependence. Yaniv's protest confused Hadar, and she said, "*We are getting confused with the definition*" [76].

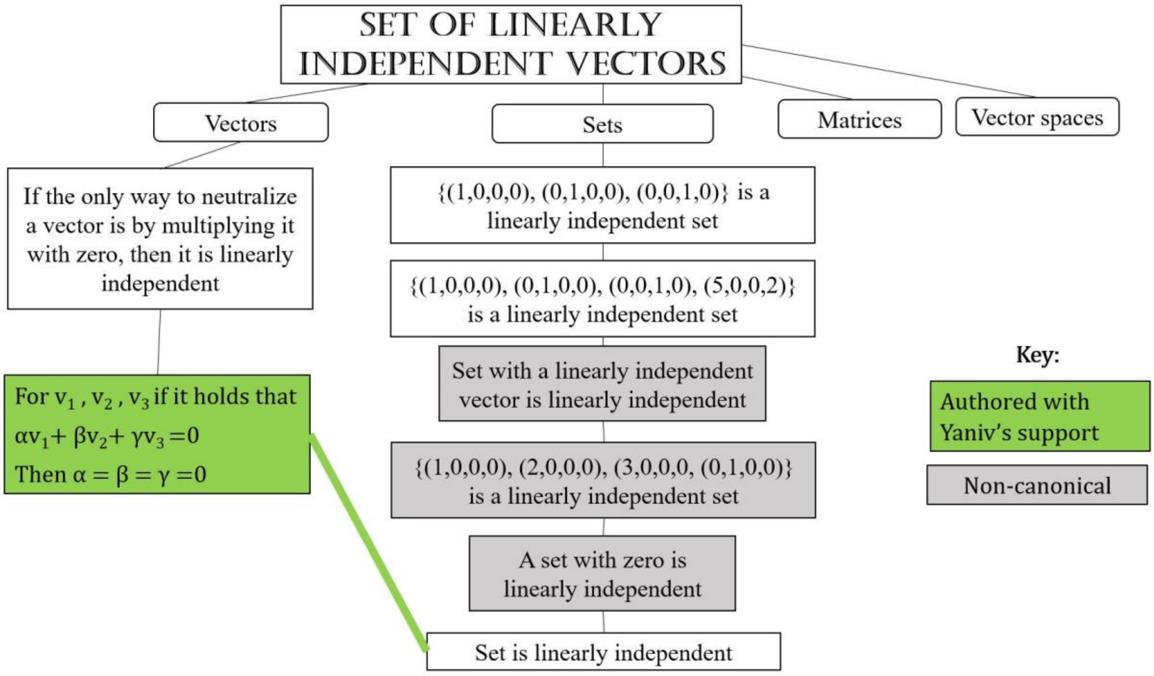

Figure 7-3 Hadar's DDMT for set of linearly independent vectors

This DDMT displays Hadar's narratives pertaining to linear independent vectors. In this case, Hadar authored narratives in both the vector subdiscourse and the sets subdiscourse. Although there are numerous narratives in the sets discourse, they did not support her use of this discourse. In this discourse she authored narratives about specific sets of vectors, which can be considered her "prototypical" dependent set. These sets include elements from the standard basis, {e_1=(1,0,0,0), e_2=(0,1,0,0), e_3=(0,0,1,0)}. The other narratives she authored in this subdiscourse were not canonical and could not support her objectification process.

In this DDMT Hadar's idiosyncratic procedure for determining if a vector is linearly independent is apparent in the vector subdiscourse. Yaniv supported her authoring an additional narrative in the vector subdiscourse. During the discussion she said, "*That means that alpha 1 is equal to alpha 2 is equal to zero, they are all equal to each other and they are equal to zero*" [72]. Yaniv told her that this means linearly independent, and she agreed by repeating his words, "*It's linearly independent*" [77]. The link between the two subdiscourses was endorsed by Hadar, but Yaniv authored it.

The two DDMTs show that the interaction with Yaniv supported Hadar authoring narratives in the vector subdiscourse and helped her author new narratives in both the vector subdiscourse and the set subdiscourse. Many of the narratives she authored in the sets subdiscourse were not canonical. She had not yet objectified the set of vectors objects, thus this subdiscourse had no meaning for her and she interpreted the narratives through the collection of vectors object. She also authored links between the subdiscourses. However, these links were with her idiosyncratic procedure and not with more general narratives. In addition, all the links to the sets subdiscourse were to the narrative "the set is in/dependent". This is a narrative she seems to be repeating ritually.

To summarize, Hadar's narratives were from within the subdiscourses of vectors and sets, and not from the coalesced discourse of sets of vectors. The objects embedded in Hadar's routine were individual vectors, whereas the narrative she was attempting to author pertained to sets of vectors. This could be seen, for example, in her statement that "*The three (vectors) are (linearly independent), but the fourth is not (linearly independent)*" [38]. The set of vectors object was still not encapsulated in Hadar's discourse, as seen when she used plural pronouns to designate the set of vectors, and not singular. This can be seen when she says, "***they** are linearly dependent*" [32], "*we will cancel **them** out*" [52], and "*we add to **them***" [54]. I conclude from this analysis that Hadar had not yet objectified the mathematical object of "sets of vectors". Hadar's incomplete objectification process meant that she was using objects from the subdiscourses of sets of vectors, and the metarules pertaining to the objects of sets of vectors discourse had no meaning for her.

7.6 Hadar and Yaniv's collaborative learning process

Yaniv's objectification process was significantly more advanced, in relation to Hadar, as far as the object "set of vectors" was concerned. Yet this did not mean that Yaniv did not have something to learn from the discussion with Hadar. On the contrary, their collaborative discussion compelled Yaniv to articulate his justifications, to connect theorems he already had endorsed to the definition and to clarify to himself the manipulations used in proofs about linear dependence. He authored new object-level narratives within subdiscourses in an endogenous development, or object-level learning. He also practiced object related metarules and developed new cross-subdiscourse narratives in horizontal exogeneous development, or meta-level learning. I hypothesize that Yaniv's location on the trajectory of objectification supported his learning on all levels and allowed him to meaningfully use the opportunity for explorative participation afforded to him in the workshop.

In contrast, Hadar was much further back in the process of objectification of the set of vectors object, and her narratives pertained to single vectors. She did author, with the support of Yaniv, new narratives within subdiscourses and authored more realizations. Thus, there was some object-level learning for Hadar. However, most of the links between subdiscourses she authored were non-canonical. The discussion between Hadar and Yaniv was ineffective in dispelling the non-canonical metarule about linear dependence being a property of single vectors that was repeatedly authored by Hadar. Although Yaniv protested against it, and once even articulated the difference in their mathematical objects, saying, "*No, it (the vector) isn't (linearly dependent) – but the set altogether is*" [43]. However, his protests were not noticed by Hadar. Moreover, the statement "the set altogether is" had no meaning in her discourse, since "the set" for her was only a collection of objects, without properties of its own. I suggest that this was the reason she did not attend to Yaniv's protests and continued to justify her claims using her idiosyncratic procedure. Thus, despite the opportunities for meta-level learning afforded to Hadar, she could not take advantage of them.

7.7 Alice and Ben – a pair with unequal participation

In the previous sections I examined the collaborative learning session of a pair of students with an egalitarian interaction. There were also groups of students where the interaction was unequal. The obstacles for learning found in Hadar and Yaniv's case, which was characterized by relatively egalitarian communicational patterns led me to hypothesize that much more serious obstacles would be found in a non-egalitarian couple. On the other hand, it could be that non-egalitarian pairs, where one student functioned as the expert and the other as novice (or follower) would have less communicational problems, and the group work would be beneficial there (at least for the novice). I turn to examine these possibilities in the next section.

As described more fully in the methods section, the initial viewing of all the recorded pairs yielded 6 recorded groups where one of the students acted as an expert and as a leader, and the other students acted as a follower. Group S3-2 (Workshop S3, Group 2) were selected for deeper analysis.

Alice and Ben, a mixed-gender pair who were previously acquainted, were two North American first year students in the Institute's International Mechanical Engineering program. They were sitting next to each other and worked together as a pair in a workshop about linear dependence. One of the reasons for choosing their collaborative session for closer analysis was based on the notes pertaining to the two students in my teaching journal. There, I wrote that after working together, the pair's presentation included Alice stating she did not agree to what she was presenting. Alice volunteered to present her and Ben's solution on the board to the class saying, "*I can try (to present our solution), but it's going to be tough*". She took Ben's notebook and started to write a proof on the board, but then said, "*How do we know this? I don't agree with what's written here*". Later she told me that she did not feel equally productive to Ben in this session. These statements of Alice's gave a first indication that the communication between the pair was not effective and that their seemingly jointly authored mathematical narrative was not endorsed by Alice.

Alice and Ben were working on the same task as Hadar and Yaniv.

> **Task**: True or false: Let V be a vector space. $\{u_1, u_2, u_3\} \subset V$ is a linearly dependent set and $u_4 \in V$ then the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent.

In contrast to Hadar and Yaniv, Alice and Ben did not author individual proofs that could be re-constructed from their initial discussion. The pair's peer-learning phase led to a non-canonical proof that was mostly authored by Ben. I first examine the patterns of communication of this pair to confirm the initial determination of an unequal interaction.

7.7.1 Channels of communication

This analysis was carried out in the same manner as the analysis of the channels of communication between Hadar and Yaniv and is exemplified on the following excerpt. The channels were labelled and colored according to the following table. In Appendix D, Section 10.4, the full analyzed transcript is available.

Private
Interpersonal Reactive
Interpersonal Proactive

The pair read the task to themselves. Ben looked only at the paper which was in front of him and started talking out loud using an instructive tone of voice.

	Speaker	**Verbal**	**NonVerbal**	**Ben's Channel**	**Alice's channel**
11	Ben	Is there a vector that you can add to this set	Looking down	Private	

		($\{u_1, u_2, u_3\}$) that will uhh…ummm..uhh..			
12	Ben	The question is – it's a combination of these ummm vectors…ummm…wait a second.	Looking down	Private	
13	Ben	Umm… It (*a vector*) is a combination.	Looking down	Private	
14	Alice	I don't know if it's (*the assertation*) always true.	Sits up suddenly		Private
15	Ben	Yeah. It (*the assertation*) is true. It is true.	Matter of fact.	Interpersonal Reactive	
16	Ben	If this ($\{u_1, u_2, u_3\}$) is linearly dependent then this ($\{u_1, u_2, u_3, u_4\}$) is linearly dependent	Looking down	Private	
17	Alice	But what if we add…			Interpersonal Proactive
18	Ben	Forget it, it doesn't matter – it's true.		Interpersonal Reactive	

Table 7-2 Classification of Alice and Ben's channels of communication

In [11] Ben restated the question asking if there could be a counter example. He wondered if there could exist a vector u_4 such that the set ($\{u_1, u_2, u_3, u_4\}$) has some property. In [12] he questioned if one of the vectors is a linear combination of the others, which is mathematically equivalent to the set being linearly dependent. However, he mentioned a specific vector "*it's*" and not "*one of the vectors*". He answered himself in [13] "*it is*". This is an outline of the proof he authored in more detail later, which is presented in the next section.

Ben did not ask Alice if she agreed with him or if she thought differently. Neither did he ask her for corroboration of any of his statements. Thus, Ben was using the private communication channel to elicit the initial narratives about the task. Initially, Alice was quiet, listening to Ben. Her first questioning of Ben's statements came in [14]. This sitting up may signal that she had an idea of her own for how to solve the task, and her stating that she was not sure it was true may hint that she had an idea of an example for which the assertation did not hold. However, this idea of an example, if ever articulated to herself, was kept in her private channel. Ben reacted to her interruption of his thoughts in the reactive interpersonal channel. In [15] he dismissed her questioning, without asking for any details of why she was objecting. He reaffirmed his statements, by simply restating the assertion, "*If this (*$\{u_1, u_2, u_3\}$*) is linearly dependent then this (*$\{u_1, u_2, u_3, u_4\}$*) is linearly dependent*" [16]. Again, his reasoning for the truth of this assertion was kept in the private channel, and again he did not ask Alice for any input.

The classification described above was carried out for all of Alice and Ben's discussion and showed large chunks where Ben's utterances were in the private channel, and he was focused on his own reasoning and ideas. Ben used the interpersonal channel almost only when responding to direct questions by Alice and requests by her for corroboration of her claims. Alice referred questions to Ben, asked him to arbitrate if a statement was correct mathematically. Ben asked Alice to "*write it neatly*", whereupon she restated his proof and turned to him to corroborate her statements using the interpersonal proactive channel. However, very few of Alice's statements are in the interpersonal reactive channel, since there was never anything for her to react to. In addition, except once at the beginning, Alice did not present any of her own ideas or reasoning. Ben's answer to her idea was to brush off her idea, without even hearing it. After this, Alice did not suggest any more of her ideas.

The few utterances of Alice that were in the interpersonal reactive channel were in places where Alice responded to a statement made by Ben. These include when Ben asked Alice questions in response to her questioning of his proof. Alice asked, "*why*?" and as part of his explanation to her, he waited for an affirmative response to his statements. There are also two places that Ben turned to Alice and asked her for corroboration of his statement. These are in the middle of chunks of Ben communicating in the private channel. There, he turned to Alice and asked for corroboration. In the first he waited until she nodded, and in the second he waited for her to respond affirmatively. He was satisfied with her response, even though she did not respond directly to his questions.

To conclude, the channels of communication analysis displayed that Ben was mostly communicating in the private channel. Alice, after once attempting to communicate in the private channel when trying to think up a counterexample to Ben's claim that the assertion was always true, communicated mostly in the proactive interpersonal channel by responding to his statements. The communication between the pair was glaringly unequal. Ben adopted the role of expert and leader, and Alice aligned herself with this.

7.7.2 Ben's DDMT

In contrast to Hadar and Yaniv, who arrived at a canonical proof for this task, Alice and Ben's discussion led to a non-canonical proof that was mostly authored by Ben. Alice hardly contributed to the pair's solution, and thus it is not possible to examine her mathematizing. I examined Ben's mathematical narratives, focusing on the subdiscourses that he used. These narratives are displayed on a DDMT in Figure 7-4, below.

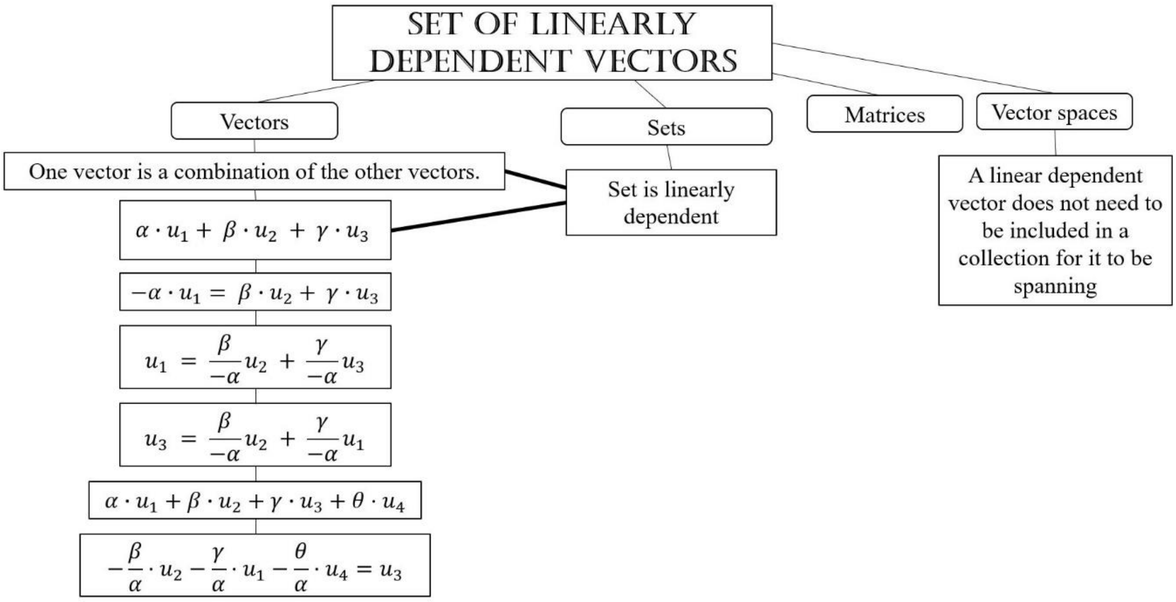

Figure 7-4 DDMT Ben Linear Dependent Set

The DDMT above shows that most of Ben's narratives were in the vector subdiscourse. Ben also authored a single narrative in the sets subdiscourse, "the set is linearly dependent" and linked it to narratives in the vector subdiscourse.

The DDMT also displays that Ben authored a single narrative in the vector space discourse. Towards the end of the pair's discussion, after Alice had restated Ben's proof and Ben confirmed that she did it correctly, Alice kept repeating "why?" and "I don't understand why?". After multiple requests, Ben attempted to answer Alice's not well-defined question with a narrative in the vector space discourse about the necessity of a linear independent vector to be able to "*span all the vectors*". He did not connect this to any other subdiscourses, and it did not satisfy Alice, who said, "*I agree with you, but I don't understand why.*" [115]. Ben finally told Alice to ask me, and that shut down the discussion. This single use of a subdiscourse, unconnected to the other subdiscourses, did not advance the pair's discussion.

7.7.3 Ben's mathematical routine for solving the task

I now examine Ben's narratives through his use of the vectors subdiscourse. In this subdiscourse there is meaning to vectors, to algebraic manipulations of vectors and to linear combinations of vectors. However, scalars have meaning in this discourse only as part of algebraic manipulations, and not as a set of scalars used in linear combinations. In the coalesced discourse of "set of vectors" the given "$\{u_1, u_2, u_3\}$ is a linearly dependent set" is the same as stating "there exists a set of three scalars, $\{\alpha, \beta, \gamma\}$, not all of them zero, such that $\alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3 = 0$". In the vectors subdiscourse a set of scalars with properties of the set (not all zero) has no meaning. Ben formed narratives using scalars as part of the narratives as arbitrary variables, without tending to their properties in the set of vectors. This can be seen in Ben's following narratives.

Ben started the proof by translating the given, "$\{u_1, u_2, u_3\}$ is a linearly dependent set" to the narrative that there exists a linear combination of the three vectors, $\alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3$. He constructed this linear combination, implicitly equated it to zero and used algebraic manipulation that led to the statement:

$$\text{(I)} \ -\alpha \cdot u_1 = \beta \cdot u_2 + \gamma \cdot u_3$$

Then Ben stated, “One of these is not zero” [23] and “one of the coefficients is not zero” [24] and continued his proof by assuming that $\alpha \neq 0$, as can be seen since he divided the equality by α, “beta over alpha, gamma over alpha” [33], to arrive at the claim:

$$(II)\ u_1 = \frac{\beta}{-\alpha}u_2 + \frac{\gamma}{-\alpha}u_3$$

In the coalesced discourse of sets of vectors, the set of scalars used in the non-trivial linear combination has a non-zero element. Ben probably interpreted this in the vectors subdiscourse as the scalar multiplying a certain vector (α multiplying u_1) is not zero. Ben next changed narrative (II) to:

$$(III)\ u_3 = \frac{\beta}{-\alpha}u_2 + \frac{\gamma}{-\alpha}u_1$$

This change was justified by him as “*I’ll change the 3 to 1. It’s usually the last one*” [35]. Changing the order of the vectors, without changing the scalars, is consistent with his use of the scalars as arbitrary variables which can change places and roles.

Ben then added a fourth vector to the linear combination he had constructed saying, “*if you add u4… then…we’re going to have*” [45] and authored the statement:

$$(IV)\ \alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3 + \theta \cdot u_4$$

He then said, “*now we want that to be equal to the zero vector in order to check whether they’re dependent or independent*” [46]. Ben’s procedure for checking if vectors are linearly dependent was to equate the linear combination constructed to zero and use algebraic manipulations to show that one vector can be written as a linear combination of the others.

He then reiterated, “*we know that alpha is not equal to zero*” [48], as within the vectors subdiscourse this is the determined value of the alpha from his previously authored narrative. Using $\alpha \neq 0$ and again switching places between the vectors u_1 and u_3, justifying it by “*instead of 3, ’cause*” [53] he authored:

$$(\text{V})\ -\frac{\beta}{\alpha} \cdot u_2 - \frac{\gamma}{\alpha} \cdot u_1 - \frac{\theta}{\alpha} \cdot u_4 = u_3$$

Ben next claimed, “This shows that these (*four vectors*) are a combination of the previous ones” [99], and thus it is linearly dependent, because “a linearly dependent set has a vector that is a combination of the others” [101]. That is, he claimed he had proved the truth of the assertion.

To summarize, Ben’s authored proof used a routine that was within the vectors subdiscourse. Alice’s minimal attempts at voicing her dissatisfaction with Ben’s set of claims, for example, asking, “*why does that (the four vectors are a linear combination equaling zero) matter*?” [100] and “*why does that (the four vectors are a linear combination) come from the definition*?” [104], were brushed off by Ben, “*It does*” [105], without providing any justification. These communication patterns of the pair which constrained any changes was also exemplified in the analysis of the patterns of communication, where Alice’s suggestion of an object level narrative was ignored.

7.7.4 Ben’s objectification process

Ben authored a proof within the vectors subdiscourse, while the property he was attempting to prove, that the set {u_1, u_2, u_3, u_4} is a linearly dependent set, pertains to a set of vectors. In this section, Ben's location on the trajectory of objectifying the "set of vectors" that was at the heart of this task is examined.

Ben used properties of vectors and manipulations on vectors to author his narratives. He constructed linear combinations, such as $(IV)\ \alpha \cdot u_1 + \beta \cdot u_2 + \gamma \cdot u_3 + \theta \cdot u_4$, by manipulating vectors, within the vectors subdiscourse. Ben added a vector to the linear combination of the three vectors, thereby constructing a linear combination of the four vectors, without considering the sets - a set with 3 elements and a set with 4 elements - as different objects. This can also be seen in the excerpt below, where he communicated in the private channel.

30	Ben:	Do we even need to do it this way? I don't even know… Fine.
31	Ben:	Beta u2 plus gamma u3. Now u1 is equal to… do we even need to do it this way? I don't even know… fine… beta over alpha gamma over alpha…(looking at paper)
32	Ben:	So, we have represented the u1 vector, in terms of a combination of the other vectors. (Looking at Alice)
33	Ben:	It shouldn't be 1 it should be three, I'll change the 3 to 1. It's usually the last one. (Changing on paper)

Ben's statements mostly related to describing manipulations of vectors. In [31] he described the construction of a linear combination and in [33] he used a procedure of changing the indices to suit what he deemed the expression should look like. His explanation aimed at Alice in [32] also describes a procedure, with the task unbonded to the original task situation of proving linear dependence.

Additionally, Ben's routines were not goal oriented, but rather procedure-oriented (or ritual). He kept trying different procedures, as can be seen in the excerpt. He used the procedure of constructing a linear combination and isolating a vector with algebraic manipulations in [31] and explains this procedure in [32]. In [33] he uses a procedure of switching between vectors. In addition, Ben's question, "*do we need to do it this way? … I don't know…fine*." [30] also indicates that he performed the algebraic manipulations for no specific goal.

Although Ben had constructed a link between the narrative "one vector is a linear combination of the others" in the vectors subdiscourse and "the set is linearly dependent" in the sets subdiscourse, he interpreted this through the vector subdiscourse. In the coalesced discourse this link is interpreted that there exists a vector which is a linear combination of the other vectors, but there is no way to determine which vector. Ben's narratives that used this link were based on a specific vector being a linear combination of the others.

To conclude, Ben's narratives pertained to the vector object, and not the "set of vectors" object, his narratives were mostly within the subdiscourse of vectors and his routine was not goal oriented, but rather a list of procedures. This leads me to suggest that he had not yet completed the objectification process for a set of vectors.

7.8 Summary of Chapter

This chapter examined the learning processes during small group discussions, where the students worked collaboratively independent from the instructor. The four students examined in this section exhibit different learning processes in a collaborative setting.

Yaniv's narratives and metarules were canonical and his objectification of the sets of vectors objects was advanced. His participation in a collaborative discussion allowed him to bond his narratives to the definition, to articulate more clearly his narratives and to construct examples and mathematical narratives justifying his claims. The interaction with Hadar supported his authoring narratives in the coalesced discourse and afforded him opportunities for explorative participation in that discourse.

Hadar had just begun the objectification process of the set of vectors object. Thus, she used metarules which were canonical in the subdiscourse of vectors, but were not canonical in the new coalesced discourse. Her participation in the collaborative discussion advanced her object-level narratives. She authored object-level narratives, examples and justifications. However, this discussion did not support the exposure of her non-canonical metarule. Thus, although the collaborative discussion advanced her object-level narratives, it did not support the meta-level shift necessary for her.

Ben's objectification process of the set of vectors object was also very preliminary, and thus he authored narratives almost exclusively in the vector subdiscourse. He participated in a peer discussion, however the discussion was not collaborative. He brushed off any challenges to his mathematical narratives, and thus did not confront alternative narratives that may have proved his ideas wrong. Moreover, he was not required (or did not feel obliged) to justify any of his statements and thus none of his erroneous claims were exposed. The discussion did not advance him at all.

Alice was not even given a chance to suggest any of her mathematical narratives. She was not given the opportunity to participate in a discussion. The peer discussion was not collaborative and did not support advancing her narratives at all.

The four students struggled independently (from an expert) with a task that had the potential for meta-level learning, as shown in Chapter 5. The main potential included object related meta-level learning by authoring narratives in the coalesced discourse, which consists of connecting between object level narratives in the separate sub-discourses available to the students. The task also had the potential for enacting executive meta-rules, such as how to prove or refute assertations. Hadar and Yaniv had difficulty with the object related metarule of linear dependence as a property of sets. Alice and Ben had difficulty with the executive meta-rule of circular logic. This difficulty with metarules occurred in both pairs of students, including the pair whose interaction was egalitarian.

The meta-level learning of connecting between object-level narratives necessitates familiarity with the object-level narratives. The tasks also had the potential for this object-level learning. The pair of students whose communicational patterns were more egalitarian took up this opportunity and advanced their object-level narratives. In contrast, the pair of students with unequal communicational patterns had difficulties with the object-level also. The students utilized the opportunity for object-level learning when the communication between them supported this. Yet, in both cases, the metarules were not sufficiently exposed and thus hindered advancement.

8 Discussion

8.1 Summary and connection to literature

There were two main goals of this study. One goal was to adapt instructional practices, shown to promote discourse-rich explorative participation to a university linear algebra course to support and encourage student participation and learning. The second goal was to explore an implementation of the above adaptation to better understand the processes of learning in an undergraduate classroom in terms of the content and the social interactions.

Active, student-centered meaningful teaching practices are discussed in many studies in elementary and secondary mathematical classrooms (e.g. Michaels et al., 2008; Schoenfeld, 2014; Smith & Stein, 2011) and in tertiary mathematical classrooms (e.g. Biggs & Tang, 2007; Hershkowitz et al., 2022; Laursen & Rasmussen, 2019; Legrand, 2001; Talbert, 2014). More instructors are aware of the importance of tertiary student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking and equitable instructional practice (Laursen & Rasmussen, 2019). This study adds to this body of literature and uses the commognitive framework (Sfard, 2008) to examine the learning processes involved in various aspects of university level mathematics (Nardi et al., 2014).

Adapting instructional practices included designing tasks aimed at promoting discourse-rich explorative participation in tertiary mathematics courses. A necessary, but not necessarily sufficient, condition for productive discussions is providing learners with tasks that support this type of setting (Cooper & Lavie, 2021). The tasks should expand students' mathematical experiences and invite students to deeper engagement (Koichu & Zazkis, 2021). I adapted the RTA tool (Weingarden et al., 2019) to design a tool to examine the designed tasks, the DMT. Using this, I found the mathematical objects embedded in the designed tasks, these objects' realizations in multiple subdiscourses, and the opportunities afforded by the tasks for saming between these realizations.

In Chapter 5, by analyzing the DMTs drawn for the designed tasks, I showed that the tasks had the potential to encourage explorative participation and to support both object-level learning and meta-level learning. The object-level learning includes the opportunities for authoring multiple realizations for mathematical objects and for saming between realizations within a subdiscourse. That is, the tasks afford opportunities for authoring narratives within a subdiscourse. Yet, the richness of the designed task is displayed by the DMT analysis of the potential for meta-level learning embedded in the tasks. The tasks afford meaningful, rich opportunities for saming realizations of linear algebra objects in different subdiscourses and for traversing the subdiscourses involved in this domain. These opportunities support the unification of different subdiscourses and the coalescing of a new discourse. Therefore, the tasks designed for the workshops had the potential for supporting explorative participation due to their capacity to provoke discussions, including compelling students to author realizations and links and providing the instructor with opportunities for highlighting unfamiliar links.

The DMT tool offers an operational definition of the potential of a task to support explorative participation. This connects to Tekkumru-Kisa and colleauges' (2020) definition of the potential of a task as the cognitive demand embedded in a task. Their definition is interwoven with the task's implementation by a teacher and how it is perceived by students. They suggest that the teacher needs to facilitate the use of the task and that the students need to have experience in solving this type of tasks in order that the cognitive demand of the task be maintained in all its phases. In contrast, the commognitive DMT tool examines the potential of a task independent of the context in which it will be used. This allows examining tasks before they are implemented in a classroom setting and supports choosing appropriate tasks for use in classrooms or textbooks.

It is also important to examine the tasks independent of the context to ensure that they can support a rich discussion. In Chapter 5 I described a commognitive analysis of possible solutions which showed that these tasks included impasses, where the student had no available routines to continue within a single subdiscourse. I showed that the solutions of tasks which support rich discussion include multiple discourses. There are many tasks that support learning, but only within a single discourse. For example, a task that asks for which values of a parameter does a given system, including a parameter, have a single solution, have infinite solutions and has no solutions. This task supports authoring realizations and practicing procedures in the matrix subdiscourse. However, it does not support the use of multiple discourses, since it can be solved completely within that discourse. Weingarden and colleagues (2019) examined classroom discussions for links authored between subdiscourses to assess explorative participation in classrooms. If a task did not have the potential for links between subdiscourses, there would be no possibility of them being authored in a classroom discussion, since a discussion facilitated by the teacher cannot include links if the potential for them does not exist. Weingarden and colleagues assumed that such links can be authored when solving the task. In this study, I did not take this assumption for granted, since one of my goals was to design the tasks and understand to what extent these indeed offer opportunities for explorative participation. I thus examined the opportunities for linking available in a task, independent of the discussion facilitated by the teacher. This demanded an extension of the methodology for constructing RTAs, as explained by Weingarden and her colleagues (2019). The DMT extended the tool to map families of objects, unlike the RTA and realization trees which use single objects as nodes. Additionally, the DMT maps the possible discourses and the links between the discourses and does not focus on the specific realizations nor on the links within discourses. Finally, the DMT maps a priori, before a discussion, what potential the task includes, whereas the RTA maps a discussion based on what was mentioned a posteriori.

This study adds to Cooper and Lavie's (2021) examination of tasks used in a lesson including explorative participation. They describe tasks that include interdiscursive use of routines and visual mediators and explain that these tasks support the students' use of a new discourse, by allowing them to draw on their precedent learners' space and the new discourse. This study adds to this and suggests that interdiscursive tasks also support linking between two subdiscourses and the use of the new, coalesced discourse. That is, interdiscursivity of a task has two facets. The first one, as Cooper and Lavie described, supports introducing students to a new, unfamiliar discourse. The second one, as described in this study, supports the coalescence of two subdiscourses into a new discourse. That is, using familiar subdiscourses to author a narrative in both. Therefore, these tasks can be used for both pedagogical aims – introducing a new discourse and for coalescing subdiscourses.

To sum up, the DMT analysis of the tasks showed that, irrespective of the context and the implementation, the tasks have the potential for both object-level learning and meta-level learning and for encouraging explorative participation. The meta-level learning embedded in the tasks includes authoring and practicing object-related metarules. This analysis also supported examining specific characteristics of these tasks, such as interdiscursivity and the inclusion of impasses.

In the second chapter of the findings (Chapter 6) I examined to what extent were the opportunities afforded by the designed tasks taken up in implemented workshops. For this, I expanded the DMT tool to a DDMT (Discussion Discourse Mapping Tree). My goal for mapping the lessons was to examine if there were realizations from within different discourses and if connections between these discourses were authored during implementations of the tasks examined in Chapter 5. The DDMT tool was designed to map subdiscourses involved in the discussions and not the specific realizations that were mentioned within each subdiscourse. While the specific realizations and the object-level narratives, from within a specific subdiscourse, are an integral part of the meta-level learning, they were not the focus of this analysis. Thus, the DDMT first utilized the DMT's a priori analysis of the subdiscourses available for the objects embedded in the task. The DDMT next utilized a posteriori analysis to draw only the realizations mentioned in class. The construction of the DDMT in this manner allowed me to map the subdiscourses which were mentioned in a discussion, which connections were authored during the discussion and who authored them. This type of analysis aligns with didactical engineering methods (Artigue, 1994) which use a priori analysis and a posteriori analysis to identify crucial phenomena and then productively implement theoretical approaches regarding this phenomena (Artigue, 2009). Artigue posits that didactical engineering methods can establish effective connections between researchers and teachers to scale up developments and disseminate pedagogical suggestions. Thus, the DMT and DDMT tools, used for research and development, might avail in the next crucial step in this project – scaling up and disseminating the developed teaching practices.

The DDMT analysis presented in Chapter 6 found that, in most cases, the implementation of the tasks included support of the students authoring narratives in multiple subdiscourses and exposing the students to links between these subdiscourses. There were numerous narratives in the new, coalesced discourse mentioned during the discussions. The students availed themselves of the opportunities provided. The mapping of the DDMT, based on viewing the recorded whole class discussions, showed that the discussions included the construction of multiple links between branches of the DDMT. The links between subdiscourses illustrate the potential for explorative participation and the potential for meta-level learning embodied in the workshop, which is authoring narratives in the new, coalesced discourse.

The analysis also showed that the opportunities for meta-level learning in the discussion were supported by the links authored or instigated by the instructor. The focus of the discussion was guided to both object-related metarules of linking between subdiscourses and executive metarules that the students were missing. The instructor also ensured that the discussion included multiple discourses, and that the discussion did not remain in a single, familiar subdiscourse. The links between the multiple subdiscourses were mostly either authored by the instructor or elicited from the students by the instructor's questions and prompts. The narratives authored by the instructor ensured that realizations were authored in multiple discourses and supported links between the discourses. This aligns with Nachlieli and Elbaum-Cohen's (2021) suggestion that student-centered instruction might support meta-level learning when strongly guided by an instructor who can explicate the new rules of the subsuming discourse and stress the limitations of the old, familiar discourse.

Another finding of this analysis was that the construction of links between subdiscourses was dependent on the students' familiarity with the narratives within the subdiscourses. This aligns with the necessity for object-level learning as a necessary precursor to meta-level learning (Sfard, 2008). Object-level learning, according to Sfard, expands an existing discourse by extending the vocabulary and producing new endorsed narratives within that discourse. Meta-level learning is usually related to a change in discourse. Sfard suggests that the change in discourse involved, that is becoming a participant in a new discourse, hinges on the capacity for using previously adopted discourses. In other words, participating in a new discourse is contingent on being familiar with the old discourse. In this study I showed that when students were not familiar with the old discourse, they did not advance to the new discourse. This occurred, for example, in the discussion about diagonalizable matrices, which was focused on procedures from the subdiscourse of eigenvalues. In contrast, when students were familiar with the old discourse, they were able to author narratives in the new, coalesced discourse. For example, in the workshop about matrices, the students were familiar with the old subdiscourses, and the discussion was very focused on linking between these.

Finally, the DDMT analysis of the whole class discussion also brought to the fore that there was usually a dominant discourse in each workshop. This dominant discourse was that which was either more familiar to the students or which included familiar procedures. Viewing the workshops with the lens of the DDMT revealed that the students authored narratives in those subdiscourses that were more familiar to them and the use of other subdiscourses had to be actively encouraged by the instructor. Additionally, the DDMT showed that students justified claims with narratives from that subdiscourse and often reverted back to using that subdiscourse, even after other, more efficient subdiscourses were introduced into the discussion. The students needed support to transition to other subdiscourses, which was a necessary step to connecting between narratives in different subdiscourses. This study extends Lithner's (2000) suggested that one of the causes of university students' difficulties in solving problems is that they focus only on the limited procedures that they remember. The limited procedures inhibit students from attempting to explore other approaches and other solutionss . This study extends this idea of the students' use of limited procedures, to the limited use of different subdiscourses. The limited procedures that the students use are probably the procedures available to them in the dominant subdiscourse. Moreover, Lavie and colleagues (2019) suggest that people interpret a task situation and thus choose a procedure within a precedent-search-space (PSS). This study expands that notion and suggests that the students choose procedures from the discourses which are within their PSS.

The DDMT analysis of the whole class discussions demonstrated various aspects of the students' participation in mathematical discussions. However, the main part of the students' independent exploration and struggle with the tasks took place in the collaborative learning phase of the workshops. During these small group discussions, the students worked independently from the instructor. Thus, I examined the learning opportunities offered to the students in the small group learning sessions.

The third chapter of the findings (Chapter 7) examined the small group learning sessions, which employed collaborative learning, to study the learning processes involved with no expert support. I examined two different types of interactions. The first was a pair of students with a mostly egalitarian interaction and a seemingly productive collaborative learning session. In this pair the commognitive analysis revealed that although one of the pair benefited from the interaction, the other did not. The second pair I examined was characterized by a glaringly unequal interaction. In this case, both the object-level learning and the meta-level learning were impaired. These findings aligns with studies that showed that unequal student identities of gender, race and the like negatively affect learning outcomes and collaboration in STEM education at university levels (Carlone & Johnson, 2007; Johnson et al., 2020).

Some evidence from previous research shows that learners' communication about the participants in the discussion may hinder their mathematical activity (Heyd-Metzuyanim & Sfard, 2012). Studying the pair with an unequal interaction showed a peer learning session with ineffectual communication in which the students did not advance in their mathematics, errors were ignored and there was no meaningful discussion. The commognitive analysis showed how the mathematics was hindered in this case, aligning with studies positing that ineffectual communication in groups might also hinder learning (Nilsson & Ryve, 2010; Sfard & Kieran, 2001). Studying Hadar and Yaniv, a pair with a mostly egalitarian interaction, showed, by analyzing the communication channels employed in the pair's interaction, that they were communicating coherently. This should support productive small group learning sessions, which needs coherent communication (Sfard & Kieran, 2001). However, although one of the learners benefited from the interaction, the other did not even though this pair was communicating coherently.

Previous studies have shown that a *commognitive conflict*, where interlocutors think they are talking about the same thing, yet in fact are using different metarules, can hinder collaborative learning (Ben-Zvi & Sfard, 2007; Sfard, 2007b, 2008). These previous studies led me to seek for the roots of Hadar's ineffective participation in the interaction in the discursive objects that the pair tended to. I did so with the aid of the analytical tools developed in Chapters 5 and 6, the DMT and the DDMT, which mapped the main challenges for the students of the workshop in terms of shifting and linking between subdiscourses. My analysis revealed that the students were discussing different objects from within different discourses and did indeed have a commognitive conflict between them. This led to difficulties in meta-level learning since the implicit metarules were not discussed, as was described by other literature (Ben-Zvi & Sfard, 2007; Chan & Sfard, 2020; Sfard, 2007b).

The analysis of the mathematical activity of the egalitarian pair showed that while the collaborative learning episode was successful for object-level learning, it did not support meta-level learning. This conclusion aligns with former claims, made in the commognitive literature, that meta-level learning requires the support of an expert attuned to the implicit metarules that the students need to learn (Nachlieli & Elbaum-Cohen, 2021). Notably, in most of the previous commognitive studies about obstacles for successful peer interactions (e.g. Chan & Sfard, 2020; Heyd-Metzuyanim & Sfard, 2012; Sfard & Kieran, 2001) the interaction was not egalitarian. For example, Sfard and Kieran (2001) describe an interaction in which Ari, the more knowledgeable partner, does not attend to his partner Gur, who in order to save face does not persist in any of his questions. This conflation between affective issues and mathematizing may have led to the conclusion that peer interactions are mostly hindered by students not listening to each other. However, my analysis of Hadar and Yaniv's interaction showed that even when the affective considerations of the interaction were optimal and supported learning, the implicit metarules of the mathematical discourse were not exposed.

The difficulty of exposing metarules in peer interaction is theorized in commognition by the idea that meta level learning requires, at least in its initial phases, ritual participation. This is due to students not being able to participate in a discourse about objects with which they are not yet familiar (Sfard, 2008). However, my analysis of the individual students showed that some students may arrive at opportunities for meta-level learning more ready than others. I found that one of the students, Yaniv, arrived at the workshop having already objectified the new mathematical object that was pertinent to the task ("set of vectors"). Thus, he was able to author narratives autonomously in the new coalesced discourse. Yaniv was seemingly on the cusp of explorative participation and the opportunity offered to him in the workshop advanced him. On the other hand, the other students, Hadar and Ben, who had not yet objectified the objects of the new discourse, benefitted only minimally from the opportunities offered by the discussion, and advanced somewhat their object-level narratives. The workshops afforded the students the opportunity for both object-level learning and meta-level learning. The students availed themselves of these opportunities in different degrees, depending on their different levels of adoption of the new discourse.

The learning opportunities offered to the students in the small group learning sessions included object-level learning and the opportunity to practice newly adopted metarules. I found, similar to what was posited by Sfard (2008), that when a student has not yet completed the objectification process of the objects in the new discourse, their narratives consider objects from the old, familiar, subsumed subdiscourse. The analysis of one student, Hadar, showed how the metarules from the new subsuming discourse, about objects from that discourse, might be used idiosyncratically. Thus, the findings in this study can help to elaborate the commognitive framework by suggesting that the meta-level learning of object related metarules hinges upon objectification in subsumed discourses. Theoretically, this study adds to the commognitive theoretical framework by suggesting how the objectification process, meta-level learning, adopting new coalesced discourses and explorative participation may be connected.

8.2 Conclusions

Designing learner-centered workshops and examining an implementation of these in a university setting allowed me to consider the productiveness and suitability of incorporating these types of non-traditional teaching methods into lectures and tutorials in university mathematics. I found that the designed workshops afforded opportunities for both object-level learning and meta-level learning, and specifically, gave the students the opportunity for explorative participation. This adds to the body of literature describing student centered teaching practices in tertiary mathematics which can benefit student learning (Griese & Kallweit, 2017; Ju & Kwon, 2007; Lahdenperä et al., 2019; Laursen et al., 2014; Laursen & Rasmussen, 2019). The whole class discussions offered the students opportunities for explorative participation. The examination of specific student's learning processes in the small group sessions showed that these did not support learning for all of the students. Learner-centered teaching in elementary schools have been studied extensively to examine how to incorporate this tool productively and successfully in mathematics classrooms (Keefer et al., 2000; Nilsson & Ryve, 2010). The assorted aspects of learner-centered teaching needs a thorough examination to incorporate it into university mathematical education in a meaningful and productive manner.

The first aspect is appropriate tasks. The tasks designed for these workshops had the potential for supporting explorative participation due to their capacity to provoke discussions, including compelling students to author realizations and links. This aligns with studies about important considerations for choosing tasks (e.g. Koichu & Zazkis, 2021; Tekkumru-Kisa et al., 2020), and extends those studies to an operational discussion of tasks that support explorative participation.

The next aspect is the instructor's role. The analysis of the workshops showed that the role of the instructor was crucial in supporting learning, especially for meta-level learning. This aligns with Michaels and colleagues (2008) suggestions for moderating meaningful mathematical discussions and extends the importance of the expert's support for the mathematical content, and not only for the socio-mathematical norms.

Another aspect is the mathematical content of the peer learning sessions. The analysis of the small group discussions showed that while collaborative learning can be productive, it is important to note what type of learning is required by the students. The peer learning sessions can be successful for object level learning. However, meta-level learning requires the support of an expert attuned to the implicit metarules that the students need to learn. Additionally, I found that successful peer learning was dependent on the compatibility of the trajectory of the objectification process of the group members. Participating in a peer discussion did not support the necessary meta-level shifts for students not advanced in the objectification process. This adds to the literature discussing the drawbacks to inquiry based and discussion-based teaching, which posit that learning without an expert, in small groups, might be arbitrary, and not advance toward the curriculum's goal (e.g. Vithal et al., 1995).

The group dynamics of the peer learning sessions, and specifically the communication between the group members, also needs to be considered. This study showed a peer learning session with ineffectual communication in which the students did not advance in their mathematics, errors were ignored and there was no meaningful discussion. The analysis showed how the mathematics was hindered in this case, aligning with studies positing that ineffectual communication in groups might also hinder learning (Nilsson & Ryve, 2010; Sfard & Kieran, 2001).

To conclude, this study showed that while inquiry-based teaching and collaborative learning can be productive, it is important to note the task, the instructor's role, the type of learning required by the students and the interactions between group members. Thus, lesson design in learner-centered teaching should be attuned to the difference between object-level learning and meta-level learning, and the teaching methods should be suited to the type of learning required.

8.3 Implications

This study has practical, methodological and empirical implications.

First, this project showed the feasibility of discussion-based linear algebra lessons and provided guiding principles for lesson design and implementation. The lesson plans and tasks designed and modified for the workshops in this project can be used by other instructors. The students' continued attendance at the workshops showed that they were interested and willing to participate in such workshops, even when it was offered in addition to the other requirements of the course. In addition, utilizing the insights from this project, tasks can be developed for other topics in the undergraduate curriculum.

Methodologically, this project developed a tool for examining the potential of tasks, the DMT, and a tool for examining the implementation of such tasks, the DDMT. These tools give an operational method of evaluating tasks by mapping the mathematical objects embedded in tasks and the available discourses. The operational definition of the potential of tasks and the operational method of examining this can be used for tasks in other mathematical topics and in other levels of mathematical education.

Empirically, this study showed the importance of noting the difference in object-level learning and meta-level learning. The peer learning sessions can be successful for object level learning. However, meta-level learning requires the support of an expert attuned to the implicit metarules that the students need to learn. This study also showed that differing levels of objectification can lead to potential commognitive conflicts among discursants.

8.4 Suggestions for future studies

This study opens up many interesting avenues of study. First, the DMT and DDMT could be studied as pedagogical tools for examining teaching and learning. The mapping tools developed for this study could be used to study other topics of university mathematics and other levels of mathematics education in similar manner. Additionally, the DMT could be used as a teaching tool in a classroom as a tool to explore mathematical objects. The students could be asked to draw a DMT for a certain object or an instructor could present a DMT to help the students visualize the connections between realizations and procedures. There are many aspects that need to be considered if one chooses to incorporate a DMT into a classroom discussion. For example, at what point in the students' learning trajectory should they be exposed to this tool and who should determine the different branches of the DMT. These intriguing possibilities need to be studied for suitability, applicability, benefits and drawbacks.

Another direction that this study opens up is the understanding of mathematical discourses and subdiscourses in tertiary classroom and the connections that need to be drawn between them for students to become fluent in these new mathematical discourses. This study mapped the subdiscourses for several mathematical objects separately. The characteristics of these discourses and their connections to discourses in other mathematical topics, such as calculus or differential equations, could be explored.

Finally, this study showed that commognitive conflicts can be due to different levels of objectification. This needs further study in different contexts, for example in K-12 classrooms, or in other university level mathematics courses. Examining interactions through the objects being discussed by the discursants might support many new possible avenues of exploration and understanding.

8.5 Limitations

The findings of this study should be examined with several limitations in mind. First, the dataset was limited. The study focused on 7 specific tasks from workshops in a linear algebra course in a specific engineering institute. The workshops were ancillary to the course, and thus were not limited by mundane considerations, such as a course syllabus and time constraints. There are many other tasks and learning goals that need to be considered when designing a course that forms a mandatory part of the undergraduate curriculum. Yet, the conclusions from this study certainly inform understanding the more general picture of university mathematics education.

Another limitation that needs to be considered is the generalizability of the findings to other mathematical topics. Linear algebra is a mathematical field where the connections between different subdiscourses are explicitly stated as a goal of the course and there are many objects with links between them. In contrast, a course about differential equations tends to be more focused on procedures of finding solutions for systems of differential equations. Tasks in this type of course might not lend themselves to mapping by DMTs, nor would DDMTs map a discussion in this course. Thus, the findings of this study are limited to university courses of a certain type. However, the main themes of this analysis can be translated to other contexts. Mapping objects in all mathematical contexts can support exploring the learning processes and the objectification process that is inherent in learning mathematics at every level. Similarly, in all mathematical contexts there is a need to differentiate between object-level learning and meta-level learning.

The examination of the in-class discussions was limited by the participants being selected by the acceptance process of the institute and had self-selected by choosing to participate in the workshops. That is, these students were self-motivated to engage in a mathematical discussion and had successfully completed previous mathematics classes. Thus, the sample of students studied in this study can certainly not be considered representative of the general population. Another limitation concerns the analysis of dyadic interactions, which was applied to only two pairs of students. There could, of course, be many other forms of interaction in the workshop that my analysis did not capture. Thus, any generalizations made from them need to be made with much caution.

Finally, the analysis of the whole class discussions focused on the opportunities for participation that were offered to the students and the narratives to which the student were exposed. As in all large group learning sessions, this analysis does not infrom us of what individual students actually learned during these sessions. Additionally, the discourse of the classroom discussion was not analyzed, and so this study cannot infrom us about the processes and the teaching-learning interaction that occurred during these discussions.

8.6 Reflections on my role as the instructor in the workshops

In this study I functioned as a participant-observer by moderating the discussion sessions, teaching regular tutorial session in the course and analyzing the data. Tabach (2011) describes how the dual role of a researcher and a teacher can enhance both roles, yet one must be aware that a teacher-researcher’s first responsibility during class is to be a teacher. During the workshops I focused on the teaching role, yet I unconsciously noted incidents that I subsequently described in my teaching journal. Thus, the researcher role was also minimally active during the workshops. The analysis of the recorded data was done in the researcher role. This enhanced my teaching practices. Stephan and Rasmussen (2002) describe how the analysis of classroom mathematics influenced their instructional practices. I agree with them that it sharpened my awareness of aspects of the theoretical approach and attention to opportunities for fostering learning. Reflection on significant events while using learner-centered methods in university mathematics classrooms supports the instructors (Nardi et al., 2005). The research role of the project necessitated my reflection on my practice and thus supported my teaching role.

Incorporating learner-centered teaching practices into habitual teaching methods is not straightforward. University mathematics educators, who agree with the importance of such teaching practices, report challenges to incorporating these methods in their classrooms.

Stewart and colleagues (2019) describe a mathematician who reverts to standard lecturing practices, while encouraging participation, due to time constraints and the need for progress according to the syllabus. These and other institutional requirements were also the source of tension for instructors altering their teaching practices to learner-centered in Mesa and colleagues' study (2020). I found that, initially, using these methods was indeed not straightforward.

Moderating a discussion-based workshop is very different from teaching traditional tutorial classes. Although my tutorial teaching style included short discussions and in-depth solutions, it is still very different from leading a workshop. During the small group phase of the workshops, the students worked by themselves, and I answered questions. This was difficult for me, since I felt that I was not teaching during that phase, as I was not being active. In addition, when students struggle it is much easier to just give them the answer, rather than pointing them in the right direction and letting them discover the answer themselves. There was a constant conflict between these, yet with more experience I found it easier to find the balance. I attempted to provide support for the students, while not giving them the final answer. Additionally, I found that answering students' questions while they were involved in struggling with the mathematics allowed me to support them in a more personalized manner.

Another difficulty for me was the absence of an exact structure and discussions planned in specific detail. The whole class discussion in the workshop was based on the questions that the students asked and the examples they constructed. Thus, although the lesson plans, written in advance, planned general ideas and families of examples, the list of these could not be exhaustive. The students kept coming up with new mistakes, new narratives and new examples. The lesson plans included difficulties students might encounter, some ways of solving the problems, and possible counter examples. However, some of the discussions were tangential to the main mathematical idea and some were based on examples and claims authored by the students. Before each workshop, I felt the stress of going into a classroom without feeling well prepared. I hoped I would be able to answer questions, think fast enough of a counter example to their claims, remember all the theorems and definitions, not get confused and not make mistakes. This is a worry teachers face when teaching for explorative particiaption (Heyd-Metzuyanim et al., 2019). However, with more experience of this type of teaching and with the support of experts with whom I could reflect on what happened during the discussions, I became more comfortable and less worried about possible issues and mistakes.

There was one workshop in which I did become confused and made a mistake on the board. In the workshop about linear transformations, I modified an example of a student and asked what changed in the properties of the linear transformation. A lively, meaningful discussion about the properties of linear transformations defined on bases ensued. Then, a student asked if the property of linearity holds for linear transformations constructed on a basis. While proving this on the board, with input from the class, I realized that the proof we were constructing was circular. We were using linearity to show that the transformation was linear. I explained to the class why there was a problem in the proof. We then discussed what needed to be proved, and what was given. First, I told the class that I would get back to them with the proof, and meanwhile we will just use the theorem. I was not comfortable with this but did not want to waste more time on it. Then I remembered that the theorem assumes linearity, and thus it cannot be proved. I explained this to the class, and the discussion continued with other topics.

At first, I felt terrible. I made a mathematical mistake on the board, and it was not an arithmetic error (which I do all the time). This was a metarule of logic and proving – we were trying to prove what needed to be assumed. I wasted valuable class time on something that did not advance the students. Watching the recording of this and receiving feedback on this incident from a mathematical pedagogical expert showed this incident in a different light. The students experienced doing real mathematics – trying things out, attempting other methods, getting confused, figuring out what was given, and deciding what needs to be proved. This was a good learning opportunity for the students during this episode, and I hope this advanced their learning. I learned that even if the worst happens – and I make mistakes - it can be used as an opportunity for teaching and learning. After I worked through it, I realized maybe it was not as terrible as I experienced it in real time. In addition, after this I was less stressed going into class. The worst had happened, and I survived, the students learned, and they were eager for the next question. But this was a stressful lesson for a teacher to learn.

The teaching aspect of the project was different from what I was used to, yet it was challenging to fathom how to adjust my teaching patterns to suit the context. It was exciting to learn new things and meaningful for me and the students.

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10 Appendices

10.1 Appendix A - Lesson Plans

10.1.1 Complex Numbers (Week 2)

Lesson Goal: The question explores logical implications between statements using complex numbers. While searching for examples for which the various statements are true, the students will practice using the definitions and representations of complex numbers. In addition, the connections between the different representations of complex numbers will be reinforced through using both representations and discussions about the complex numbers, thus bolstering the objectifying of the complex field and its realizations. The question also examines the logic involved, by discussing when two statements are equivalent, when one implies the other and when there is no connection between them.

Introduction: (7 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

1. Algebraic representation (a+ib), Trigonometric representation ($r \cdot \text{cis}\,\theta$), Geometric representation (2-dimensional plane)
2. Im(z), Re(z), $\bar{z}$, |z|, arg(z)

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

Let $z_1, z_2 \in \mathbb{C}$ such that $z_1, z_2 \neq 0$.

1) Let $z_1 \cdot z_2 \in \mathbb{R}$. Which of the following statements is always true? Which statement is never true? Which statement holds for specific cases of $z_1, z_2 \in \mathbb{C}$?

    a) $z_1 = \overline{z_2}$
    b) $z_1 = \alpha \cdot z_2$ ($\alpha \in \mathbb{R}$)
    c) $z_1^2 \cdot z_2^2 = 1$
    d) Im (z_1) = 0

2) Give a statement for which the following is true:

$$\frac{z_1}{z_2} \in R \iff \text{(statement)}$$

Solutions:

1) $z_1 \cdot z_2 \in \mathbb{R}$
    a) Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (a) holds:

$z_1 = 1 + i,\ z_2 = 1 - i = \overline{z_1}$ and $(1 + i) \cdot (1 - i) = |z_1|^2 = 2 \in \mathbb{R}$

Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (a) does not hold:

$z_1 = 1 + i,\ z_2 = 3 - 3i = 3 \cdot \overline{z_1} \neq \overline{z_1}$ and $(1 + i) \cdot (3 - 3i) = 6 = 3 \cdot |z_1|^2 \in \mathbb{R}$

    b) Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (b) holds:

$$z_1 = i,\ z_2 = 2i,\ z_1 = \frac{1}{2} \cdot z_2, \frac{1}{2} \in R \text{ and } z_1 \cdot z_2 = -2 \in \mathbb{R}$$

Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (b) does not hold:

$z_1 = 1 + i, z_2 = 2 - 2i, z_1 \neq \alpha \cdot z_2, \quad \alpha \in \mathbb{R}$ and
$$z_1 \cdot z_2 = (1 + i) \cdot 2 \cdot (1 - i) = 2 \cdot |1 + i|^2 = 4 \in \mathbb{R}$$

c) Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (c) holds:
$z_1 = i,\ z_2 = -i,\ z_1^2 \cdot z_2^2 = i^2 \cdot (-i)^2 = (-1) \cdot (-1) = 1$ and $z_1 \cdot z_2 = 1 \in \mathbb{R}$

Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (c) does not hold:
$z_1 = i, z_2 = 3i, z_1^2 \cdot z_2^2 = i^2 \cdot (3i)^2 = (-1) \cdot (-9) \neq 1$ and $z_1 \cdot z_2 = -3 \in \mathbb{R}$

d) Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (d) holds:

$z_1 = 1,\ z_2 = 2\ ,\ \ z_1 \cdot z_2 = 1 \cdot 2 = 2 \in \mathbb{R}$ and $\operatorname{Im}(z_1) = 0$

Example of $z_1 \cdot z_2 \in \mathbb{R}$ and (d) does not hold:
$z_1 = i,\ z_2 = i$ and $z_1 \cdot z_2 = i \cdot i = -1 \in \mathbb{R}$ but $\operatorname{Im}(z_1) = 1 \neq 0$

2) Statements that are equivalent to $\frac{z_1}{z_2} \in \mathbb{R}$:

Geometric representation:

$$\frac{z_1}{z_2} \in \mathbb{R} \Leftrightarrow \frac{r_1 cis\theta_1}{r_2 cis\theta_2} \in \mathbb{R} \Leftrightarrow$$
$$\frac{r_1}{r_2} \cdot cis(\theta_1 - \theta_2) \in \mathbb{R} \Leftrightarrow \operatorname{Im}(\frac{r_1}{r_2} \cdot cis(\theta_1 - \theta_2)) = 0 \Leftrightarrow$$
$$\sin(\theta_1 - \theta_2) = 0 \Leftrightarrow \theta_1 - \theta_2 = \pi \cdot k, k \in \mathbb{Z}$$
$$\Leftrightarrow \theta_1 = \theta_2 \text{ or } \theta_1 = \pi + \theta_2 \text{ or } \theta_2 = \pi + \theta_1$$
$$\Leftrightarrow arg(z_1) = arg(z_2) \text{ or } arg(z_{1,2}) = \pi + arg(z_{2,1})$$

Algebraic representation:

$$\frac{z_1}{z_2} \in \mathbb{R} \Leftrightarrow \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \in \mathbb{R} \Leftrightarrow \frac{(a_1 + ib_1) \cdot (a_2 - ib_2)}{a_2^2 + b_2^2} \in \mathbb{R} \Leftrightarrow$$
$$(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2) \in \mathbb{R} \Leftrightarrow$$
$$a_2 b_1 - a_1 b_2 = 0 \Leftrightarrow$$

If $a_1, a_2 \neq 0$ $\frac{b_2}{a_2} = \frac{b_1}{a_1}$ $\Leftrightarrow$ arg? θ? geometric?

If $a_1 = 0 \Rightarrow b_1 \neq 0$ since $z_1 \neq 0$ and then $a_2 b_1 - a_1 b_2 = 0 \Rightarrow$
$a_2 b_1 = 0 \Rightarrow b_2 = 0$ which means $z_1 = b_1 \cdot i, z_2 = b_2 \cdot i$

and so in this case ($a_1 = 0$) $\frac{z_1}{z_2} = \frac{b_1 \cdot i}{b_2 \cdot i} = \frac{b_1}{b_2} \in \mathbb{R}$.

Connection between representations:

- $arg(z_1) = tg^{-1}\left(\frac{b_1}{a_1}\right)$

- $tg(\theta_1) = -tg(\theta_2) \Leftrightarrow \theta_1 + \theta_2 = \pi k$ or $\theta_1 = -\theta_2$

- Drawing the numbers on a Complex plane

General discussion (15 minutes) The students will be asked to present on the board:

1) Examples of complex numbers for which the statements are true and to show this.
2) Examples of complex numbers for which the statements are false and to show this.
3) Discussion are any of the given statements equivalent to $z_1 \cdot z_2 \in \mathbb{R}$.
4) Examples of statements that are equivalent to $\frac{z_1}{z_2} \in \mathbb{R}$.

The connections between the different representations of complex numbers will be indicated. The students can be asked if the solutions are the same and how do they connect.

Questions for further discussion

These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is extra time in class, they can be discussed with the whole class.

1) Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n such that $a_i \in \mathbb{R}$.

Prove that if n is odd then $p(x)$ has a real root.

2) Is the other direction true? Is n odd $\Leftrightarrow$ $p(x)$ has a real root?

Conclusion (5 minutes) The connections between the different representations of Complex numbers will be reified, the Complex field and its elements as a mathematical object will be discussed. In addition, when two statements are equivalent, when one can be induced from the other but not the other way will be mentioned.

10.1.2 Matrices (Week 3)

Lesson Goal: The question explores how the rows of C influence the rows of CD, and how the columns of D influence the columns of CD. The question also demonstrates that the opposite does not hold, that is that the columns of C do not affect the columns of CD and the rows of D do not affect the rows of CD.

In addition, the question practices manipulating matrices in different ways - as arrays of numbers, as sets of rows, $n \times n$ elements and as a mathematical objects. The connections between these different methods of representing matrices furthers the objectification of the concept of matrices.

Introduction: (7 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

- A_{45} is element in fourth row, and fifth column
- Matrix Multiplication: $AB_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$
- $A_{ij}^{t} = A_{ji}$, Symmetric and Anti-symmetric matrices

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

Let C be a matrix whose third column is all zero's.

Let D be a matrix whose second row is all zero's.

Examine CD and DC. Do they inherit any characteristics from C and D? That is, is the third column all zero's? Is the second row all zero's?

Solution Method	**Possible Difficulties**	**Advancing Questions**
Trying on numerical examples $C \cdot D = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ $= (*)$	Finding minimal dimensions for the question to be well defined. Multiplying matrices wrong Columns and rows mixed up Order of multiplication wrong	How do you multiply matrices? Where is column 3? Is $CD = DC$?

$D \cdot C = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 5 & 9 & 0 & 6 \\ 6 & 9 & 0 & 6 \end{pmatrix}$		
Characterization of matrix $\forall i \;\; C_{i3} = 0$ $\forall j \;\; D_{2j} = 0$	Wrong definition Order wrong	Write out an example Show me element 4,3 of the matrix
Picture		
Calculating elements $(CD)_{2j} = \sum_{k=1}^{n} C_{2k} D_{kj}$		

General discussion (15 minutes) The students will be asked to present their solution on the board in the following order:

1. Some worked examples
2. Picture
3. Calculating elements
4. General matrix

This order follows the order that the students' understanding of the concept usually takes, and so builds on their previous understanding of concrete objects - arrays of numbers - to scaffold their understanding of a matrix as an object that can be manipulated as a unit.

The connections between the different representations will be pointed out. The students can be asked if the solutions are the same and how do they connect.

Questions for further discussion (8 minutes)

These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

1. If only one of the matrices has such a characteristic, does the product still have it?

The question is given that both C and D have certain traits, is this necessary? This can lead to the discussion what is the minimal conditions necessary, so that a zero appears in the product matrix. If only one element in the matrix is zero, what effect will this have on the product? Can they construct two matrices with no zero entries such that the product will have a row of zeros?

2. What other such traits are conserved by matrix multiplication?

This question is also a review of other new concepts, such as symmetric matrices, anti-symmetric matrices, scalar matrices. The students can also define their own traits, such as a row of all ones, all the elements in the given matrices are positive, all the elements are whole numbers.

Conclusion (5 minutes) The connections between the elements of the matrix, the matrix as an array and a matrix as a mathematical object will be pointed out.

10.1.3 Systems of Linear Equations (SLE) (Week 4)

Lesson Goal: The question explores Systems of Linear Equations. These systems can be represented as a list of equations, as a matrix and as a list of constraints on the variables. The question asks for examples of systems, thus the student will explore the connections between these different representations and this will support objectification of SLE's.

SLE is an obvious use of matrices, and students appreciate the matrices they just learned when they see it as a useful and powerful tool. The set of solutions of a homogenous system is a vector space, and thus while learning vector spaces the students have a tangible example to project the concept on.

Introduction: (7 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

- SLE is a system of n linear equations with m variables.
- A SLE can be written in matrix form $A \cdot X = b$, $A \in F^{n \times m}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$.
- A SLE has no solution iff $r(A|b) \neq r(A)$;
  A SLE has exactly one solution iff $r(A) = r(A|b) = m$;
  A SLE has infinite solutions iff $r(A) = r(A|b) < m$.

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

$2x - 3y + z = 4$ is a linear equation with 3 variables.

a) Give a system of linear equations, including the one above, such that there will be no solution to the system; there will be exactly one solution to the system; there will be an infinite number of solutions to the system.
b) Give a system of linear equations, including the one above, such that (1,2,8) will be a solution to the system AND there will be exactly one solution to the system; AND there will be an infinite number of solutions to the system.

Possible Solution

No solution:

(a) $\begin{cases} 2x - 3y + z = 4 \\ 2x - 3y + z = 5 \end{cases}$

For (b) there is no such system, since if there exists a solution, than there is not a case of no solution.

Possible difficulty: Confusing single solution, does not exist any other solution, with no solution, does not exist any solution at all

Advancing Question: How many solutions are there for system?

Single solution:
The following matrix has a rank of 3, like the number of variables, so the system it represents has a single solution.

(a) $\begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The system represented is $\begin{cases} 2x-3y+z=4 \\ y=0 \\ z=0 \end{cases}$

For (b) we first check that (1,2,8) is a solution of the equation given : $2\cdot(1)-3\cdot(2)+(8)=4$
It is.

Then we build 2 more equations that this is the only solution possible: $y=2$ and $z=8$. If we do not give two more equations, than there will be more variables than equations and we can find more than one solution. A less elegant solution : $y+z=10$ and $x+z=9$. There are 3 constraints on the variables and the only solution is the one given.

Possible difficulty: Giving 3 equations when they are dependent.
Advancing question: Ask student to solve system and then discuss why the 3 equations are the same information (multiple of each other etc.)

Infinite solutions
$\{2x-3y+z=4$ OR $\begin{cases} 2x-3y+z=4 \\ 4x-6y+2z=8 \end{cases}$ OR $\begin{cases} 2x-3y+z=4 \\ -2x+3y-z=-4 \end{cases}$

For (b), since we saw from above that (1,2,8) is a solution, than any system given in (a) is also a solution for (b).

Possible difficulty: Constructing a new system could lead to arithmetic errors or to contradictions, which would give no solutions.
Advancing Question: What constraints are necessary to have infinite solutions?

General discussion (15 minutes) The students will be asked to present their solution on the board for each case for (a) and for (b).

1) No solution
2) Single solution
3) Infinite solutions

The connections between the different representations will be pointed out. The students can be asked if the solutions are the same and how do they connect.

Questions for further discussion (8 minutes)

These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

1) What is the largest number of equations that can be used as an example that answers the question?
2) What is the smallest number of equations that can be used as an example that answers the question?
3) Ask the students for a set of constraints on the vectors, for example $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$ and ask the students to build a system that this is the solution.

Conclusion (5 minutes) The connections between the matrix representing the SLE and the equations will be stressed. In addition, the rank of the matrix gives an indication of how many equations are the minimum necessary for solving the question.

10.1.4 Subspace (Week 6)

Lesson Goal: The question examines subspaces of $\mathbb{R}^{2\times 3}$. In order to answer the question, the students should build examples of subspaces that fulfil the requirement, and thus they have an opportunity to objectify subspaces and the connections between them. The question focuses on a specific Vector Space in order to scaffold the objectifying of subspaces,
and the question also asks for a maximal value of n, thus answering the question should lead to a discussion of when examples are a sufficient proof of a concept and how can a maximal value be proved? Does existence of an example for a specific value of n suffice for a proof?

Introduction: (7 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

- A **vector space** is a set of vectors and a field of scalars for whom the list of 10 properties hold.

To prove a set of vectors, with scalars, is a V.S. 10 properties need to be examined.

- A **subspace** is a non-empty subset of vectors that is closed under addition and scalar multiplication, for the same field.

To prove a subset of vectors is a subspace 3 properties need to be examined. (The three can be combined into 2.)

- Let $U, W \subseteq V$ be subspaces of a vector space V. Then $U \cap W$ and $U + W$ are also subspaces. $U \cup W$ is a subspace iff $U \subseteq W$ or $W \subseteq U$.

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

What is the greatest value of n, such that there exist subspaces W_i, $1 \le i \le n$ of $\mathbb{R}^{2\times 3}$ such that:

$$W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{n-1} \subsetneq W_n$$

Solution:
Example of subspaces:

Example 1:

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \subsetneq \left\{\begin{pmatrix} a & a & a \\ a & a & a \end{pmatrix} \middle| a \in \mathbb{R}\right\} \subsetneq \left\{\begin{pmatrix} a & a & a \\ b & b & b \end{pmatrix} \middle| a, b \in \mathbb{R}\right\} \subsetneq \mathbb{R}^{2\times 3}$$

This example is for $n = 4$, it can be expanded for a greater value of n.

Example 2:

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \subsetneq \left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R}\right\} \subsetneq$$

$$\subsetneq \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b \in \mathbb{R} \right\} \subsetneq \cdots \subsetneq \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \middle| a,b,c,d,e,f \in \mathbb{R} \right\} = \mathbb{R}^{2\times 3}$$

In this example $n=7$.

General discussion (15 minutes)
The students will be asked to present their solution on the board.

For each example the students give, they will be asked to prove on the board that each W_i is a subspace and that they are not equal.

1. How does choosing a different order of putting in parameters change the Example 2?
2. How can Example 1 be expanded?
3. Given the Theorem: For subspaces $S,T \subset V$:

   $$S \subsetneq T \Leftrightarrow \dim S < \dim T$$

   How does this effect the answer to the question.

Questions for further discussion
These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

1. Using the 7 different subspaces from above, how many different subspaces can be constructed using intersection, union and sum?
2. For which k is $W_k + W_{k+1} = W_n$?
3. Give example of $U,W \subsetneq V$ subspaces, such that $U + W = V$.
4. Do there exist subspaces for which the sum is direct $W_k \oplus W_{k+1} = W_n$?
5. Construct t subspaces of $\mathbb{R}^{2\times 3}$ such that $W_1 \oplus W_2 \oplus \cdots \oplus W_t = \mathbb{R}^{2\times 3}$.
6. What is the maximal /minimal t?

<u>Possible Solutions</u>

1. If $W_i = \{\vec{0}\}$ or $W_i = V$, then union and intersection is:

$W_j \cap V = W_j$, $W_j \cap \{\vec{0}\} = \{\vec{0}\}$, $W_j \cup \{\vec{0}\} = W_j$, $W_j \cup V = V$

If $W_i \subsetneq W_j$ then:

$W_i \cap W_j = W_i$ and $W_i \cup W_j = W_j$

So no new subspaces will be constructed.

2. If $W_i \subsetneq W_j$ then: $W_i + W_j = W_j$, so only for $k=6$ is this statement true.
3. For $U = \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$ and $W = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

   $U + W = V$ and since $U \cap W = \{\vec{0}\}$ then (4) $U \oplus W = V$

5. $U_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$

$$U_2 = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

$$U_3 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

$$U_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

$$U_5 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

$$U_6 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

$$U_7 = \{\vec{0}\}$$

6. Maximal t: t = 7
   Proof: Using Dimension theorem: $dim(U+W) = dimU + dimW - dim(U \cap W)$
   If sum is direct, then $dim(U \cap W) = 0$
   Minimal t: (t = 1 fulfils conditions, but is not interesting) t=2:
   Proof: $V \oplus \{\vec{0}\} = V$
   Using $\{\vec{0}\} \neq W, U \subsetneq V$:

$$U = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\} \text{ and } W = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$$

$U \cap W = \{\vec{0}\}$ and $U + W = V$ so $U \oplus W = V$

Conclusion (5 minutes)

The connections between the dimension of the vector space and the possible dimensions of the subspaces will be discussed. The concept of subspaces can be generalized to a general vector space V of dimension n. Objectifying the concrete vector space (matrices of a specific order) should expedite the objectifying of a theoretical general vector space.

10.1.5 Linear Dependence (Week 7)

Lesson Goal: The question explores linear dependence between vectors and the spanning space of a set of vectors. Constructing examples allows students to investigate when vectors are linearly dependent, which promotes understanding of the connection between the linear span and a minimal spanning set. This is a necessary foundation for understanding basis and dimension of a subspace, which is the next topic in the course.

In addition, solving the question utilizes counter examples to prove a statement. Understanding the logic when a counter example constitutes a proof and when it is not sufficient is difficult for students. The discussion about the answer to the question could be guided to discussions about what is a sufficient proof? When is a statement always true, when is it sometimes true, and when is it never true? and when is a counter example sufficient.

The typical example given as an answer is from $\mathbb{R}^n$, as this is what the term "vectors" denominated in high school. If during the discussion no other examples are given, then advancing questions will be asked in order that examples of matrices, vectors and polynomial vectors will be introduced. Manipulating elements from these vector spaces, i.e. matrices and polynomials, will help students objectify those elements as vectors also.

Complex fields add another level of complexity to the question, thus are left for the end of the discussion. If a student has difficulty manipulating real matrices, then complicating it with complex numbers does not help them. However, after the concept of matrices is objectified, using complex numbers as entries in the array is straightforward.

Introduction: (5 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

A set of vectors $\{v_1,\ldots,v_n\}\subseteq V$ is *linearly dependent* over F (a field of scalars) in $\mathbb{V}$ (a vector space), if there exist scalars $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, not all zero, such that $\sum_{i=1}^{n}\alpha_i\cdot v_i=\vec{0}$.

If the only scalars for which $\sum_{i=1}^{n}\alpha_i\cdot v_i=\vec{0}$ are $\alpha_1=\alpha_2=\cdots=\alpha_n=0$, then the vectors are *linearly independent.*

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

$\mathbb{V}$ is a vector space over the field F. Are the following statements True or False?

If a statement is true, prove it. If a statement is false, give a numerical counter example.

1. $\{u_1,u_2,u_3\}\subseteq V$ is a linearly independent set and $u_4\in V$, then the set $\{u_1,u_2,u_3,u_4\}$ is linearly independent.

2. $\{u_1,u_2,u_3\}\subseteq V$ is a linearly dependent set and $u_4\in V$, then the set $\{u_1,u_2,u_3,u_4\}$ is linearly dependent.

3. If $\{u_1,u_2,\ldots u_6\}\subseteq V$ is a linearly dependent set, then $Sp\{u_1,...,u_5\}=Sp\{u_2,...,u_6\}$.
4. If $\{u_1,u_2,\ldots u_6\}\subseteq V$ is a linearly independent set, then $Sp\{u_1,...,u_5\}=Sp\{u_2,...,u_6\}$

Possible Solution

1. False. Counter example: For $V=\mathbb{R}^4$, then
   $u_1=(1,0,0,0), u_2=(0,1,0,0), u_3=(0,0,1,0)$ are linearly independent,
   $u_4=(1,1,1,0)\in V$ and $\{u_1,u_2,u_3,u_4\}$ is a linearly dependent set.
   Using the definition : choose $\alpha_1=1,\alpha_2=1,\alpha_3=1,\alpha_4=-1$,
   Or using a result that u_4 is a linear combination of the other 3.
   Difficulties: Confusing definition of linear independence and dependence.
   Advancing question: What is the definition?
   Note: Exists u_4 so that $\{u_1,u_2,u_3,u_4\}$ is a linearly independent set, for example $u_4=(0,0,0,1)$. The above is a counter example, even though there exists examples when it is true.
2. True.
   Proof: If $\{u_1,u_2,u_3\}$ is linearly dependent, then there exist $\alpha_1,\alpha_2,\alpha_3$ not all zero, such that $\alpha_1\cdot u_1+\alpha_2\cdot u_2+\alpha_3\cdot u_3=\vec{0}$. Let $\alpha_4=0$, and then there exist $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, not all zero, such that $\alpha_1\cdot u_1+\alpha_2\cdot u_2+\alpha_3\cdot u_3+\alpha_4\cdot u_4=\vec{0}$.
   That is : $\{u_1,u_2,u_3,u_4\}$ is a linearly dependent set.
   Difficulties: Confusing definition of linear independence and dependence.
   Advancing question: What is the definition?

3. Example for yes:

$\{(1,0,0),(0,1,0),(2,0,0)\}$ a linearly dependent set

$$Sp\{u_1,...,u_{n-1}\}=Sp\{(1,0,0),(0,1,0)\}=\{(x,y,0)\,|\,x,y\in\mathbb{R}\}$$

$$Sp\{u_2,...,u_n\}=Sp\{(0,1,0),(2,0,0)\}=\{(2t,y,0)\,|\,t,y\in\mathbb{R}\}=\{(x,y,0)\,|\,x,y\in\mathbb{R}\}$$

Example for no:

$\{(1,0,0),(2,0,0),(0,1,0)\}$ a linearly dependent set

$$Sp\{u_1,...,u_{n-1}\}=Sp\{(1,0,0),(2,0,0)\}=\{(x+2y,0,0)\,|\,x,y\in\mathbb{R}\}=\{(x,0,0)\,|\,x\in\mathbb{R}\}$$

$$Sp\{u_2,...,u_n\}=Sp\{(2,0,0),(0,1,0)\}=\{(2x,y,0)\,|\,x,y\in\mathbb{R}\}=\{(t,y,0)\,|\,t,y\in\mathbb{R}\}$$

$Sp\{u_1,...,u_5\} \neq Sp\{u_2,...,u_6\}$ since $(1,2,0) \notin Sp\{u_1,...,u_5\}$ but $(1,2,0) \in Sp\{u_2,...,u_6\}$

So statement is false, since exists a counter example.

However, there exists examples when the statement is true.

4. False, Counter example:

$\{(1,0,0),(0,1,0),(0,0,1)\}$ a linearly independent set

$Sp\{u_1,...,u_{n-1}\} = Sp\{(1,0,0),(0,1,0)\} = \{(x,y,0) \mid x,y \in \mathbb{R}\}$

$Sp\{u_2,...,u_n\} = Sp\{(0,1,0),(0,0,1)\} = \{(0,x,y) \mid x,y \in \mathbb{R}\}$

$Sp\{u_1,...,u_5\} \neq Sp\{u_2,...,u_6\}$

Proof that it is never true, that is always $Sp\{u_1,...,u_5\} \neq Sp\{u_2,...,u_6\}$:

Let $\{u_1,u_2,\ldots u_n\} \subseteq V$ be a *linearly independent* set, such that $Sp\{u_1,...,u_{n-1}\} = Sp\{u_2,...,u_n\}$.

Then $u_n \in Sp\{u_1,...,u_{n-1}\}$, so there exist scalars $\alpha_1,\ldots,\alpha_{n-1}$ such that $u_n = \sum_{i=1}^{n-1} \alpha_i \cdot v_i$

$\{u_1,u_2,\ldots u_n\}$ is linearly independent, so $\vec{0} \notin \{u_1,u_2,\ldots u_n\}$, that means $u_n \neq \vec{0}$, so

not all α_i, $1 \leq i \leq n-1$ are zero.

Thus, $-u_n + \sum_{i=1}^{n-1} \alpha_i \cdot v_i = \vec{0}$, where not all the scalars are zero, then $\{u_1,\ldots,u_n\}$ is a linear dependent set, which is a contradiction to the given.

So, $Sp\{u_1,...,u_{n-1}\} \neq Sp\{u_2,...,u_n\}$

General discussion (15 minutes) The students will be asked to present their examples on the board in the following order:

1. $\mathbb{R}^n$
2. $\mathbb{R}^{n\times n}$
3. $\mathbb{R}_n[x]$

This order starts with the familiar vectors and then shows other vector spaces.

The connections between the different vector spaces will be pointed out. The students can be asked if the solutions are the same and how do they connect.

Questions for further discussion} (10 minutes)

These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

1. What if $V=\mathbb{R}^3$? Will (1) and (2) change?
2. For which V will (3) and (4) change?
3. Examples utilizing complex numbers:
   i) For $V=\mathbb{C}^{3\times 3}$ give an example of vectors for whom the statement is true and when the statement is false.
   ii) For $V=\mathbb{C}_n[x]=\{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0 \mid a_i\in\mathbb{C}\}$

Conclusion (5 minutes)

We will review the main concepts:

1) A counter example proves that the opposite statement is true, but does not prove that the statement is always false.
2) An example does not prove a statement is always true.
3) Linear span of different sets of vectors can be equal.

10.1.6 Linear Transformations (Week 10 or 11)

Lesson Goal: This question examines the relationship between a linear transformation and the dimension of its kernel. Constructing examples of transformations that conform to the definition given illustrates this for the student.

Linear transformations can be defined in various ways, and thus the question can be solved using any of the definitions. However, it is simpler to use the appropriate definition for the different parts. This question allows the student to consider which method is more efficient for use, and highlights the connections between the different definitions.

Introduction: (7 minutes)
Reminding the students of the basic definitions that they saw in class.
Definitions:

1. $T: V \rightarrow W$, where V and W are vector spaces, is a Linear Transformation if
$$\forall \vec{v}, \vec{w} \in V,\ \forall \alpha \in F\ \ T(\alpha \cdot \vec{v} + \vec{w}) = \alpha \cdot T(\vec{v}) + T(\vec{w})$$
2. Ker (T) = $\{\vec{v} \in V \mid T(v) = \vec{0}\}$
3. Im(T) = $\{ T(\vec{v}) \in W \mid \vec{v} \in V \}$

Question: (15 minutes)
The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

Question:
Let T: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that T(1,2,3) = (0,0,0)
and T is not the zero transformation.

1. Give an example of such a T such that dimKer T = 0, if there exists such a transformation. Find a basis for Ker T and a basis for Im T.
2. Give an example of such a T such that dim Ker T = 1, if there exists such a transformation. Find a basis for Ker T and a basis for Im T.
3. Give an example of such a T such that dim Ker T = 2, if there exists such a transformation. Find a basis for Ker T and a basis for Im T.
4. Give an example of such a T such that dim Ker T = 3, if there exists such a transformation. Find a basis for Ker T and a basis for Im T.

Possible Solution

1. (1,2,3) is in Ker T, so dim Ker T $\geq$ 1. Does not exist such a T.
2. Complete {(1,2,3)} to a basis for $\mathbb{R}^3$: {(1,2,3),(0,1,0),(0,0,1)}
Define:
$$T(1,2,3) = (0,0,0)$$
$$T(0,1,0) = (1,0,0)$$
$$T(0,0,1) = (0,1,0)$$

Thus T(x,y,z) = T((x) · (1,2,3) + (y-2x) · (0,1,0) + (z-3x) · (0,0,1)) =
= (x) · (0,0,0)+(y-2x) · (1,0,0)+(z-3x) · (0,1,0) = (y-2x,z-3x,0)

B_{KerT} = {(1,2,3)} so dim Ker T = 1

$B_{Im\,T}$ = {(1,0,0) , (0,1,0)} so dim Im T = 2

Using Solution method: Spanning set for Im T is {T(v_1), T(v_2),T(v_3)} where {v_1, v_2, v_3} is a basis.

dim Ker T + dim Im T = dim V = dim $\mathbb{R}^3$

3. Complete to a basis and define:

$$T(1,2,3) = (0,0,0)$$
$$T(0,1,0) = (0,0,0)$$
$$T(0,0,1) = (1,1,1)$$

Thus $T(x,y,z) = T((x) \cdot (1,2,3) + (y-2x) \cdot (0,1,0) + (z-3x) \cdot (0,0,1)) =$
$= (x) \cdot (0,0,0) + (y-2x) \cdot (0,0,0) + (z-3x) \cdot (1,1,1) = (z-3x,z-3x,z-3x)$

$B_{KerT} = \{(1,2,3), (0,1,0)\}$ so dim Ker T = 2
$B_{Im\ T} = \{(1,1,1)\}$ so dim Im T =1

Using Solution Method: Im T = $\{(z-3x, z-3x,z-3x) \mid z,x \in \mathbb{R}\}= \{(z-3x) \cdot (1,1,1) \mid z-3x \in \mathbb{R}\} = \{\alpha \cdot (1,1,1) \mid \alpha \in \mathbb{R}\} = Sp\{(1,1,1)\}$ so $B_{Im\ T}= \{(1,1,1)\}$ since a single spanning vector is linearly independent.

4. If dimKer T = 3 then dim Im T = 0, so T = 0. However, it is given that T is not the zero transformation. So such a T does not exist.
   OR
   If dim Ker T = 3 Then Ker T is a subspace of $\mathbb{R}^3$ with dimension 3, so Ker T = $\mathbb{R}^3$, which means that T is the zero transformation.

Part of Solution	**Possible Difficulties**	**Advancing Questions**
Completing to a basis of $\mathbb{R}^3$ {(1,2,3) , (0,1,0) , (0,0,1)}	Choosing a linear dependent set	Why is it a basis? When is a set a basis?
	Choosing non-spanning set	What is the general element of $\mathbb{R}^3$?
	Not using a basis to define T	**
Defining T on this basis	Not linear	What is T(0,0,0) or image of sum of some vectors?
	Does not fulfil condition of question	What is T(1,2,3)?
	Dimension of Ker T is wrong	Which vectors are in Ker T?
Finding general element	Finding scalars	
Finding bases		
Using Dimension Theorem to prove 3 not possible		What is Im T, if T is zero vector?

General discussion (15 minutes)

The students will be asked to present their solution on the board in the order of the questions.

Questions for further discussion (8 minutes)

This question can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

Redefine T such that T(1,2,3)=(1,2,3), answer the same questions as above.

1. dim Ker T = 0

$$T(1,2,3) = (1,2,3)$$
$$T(0,1,0) = (0,1,0)$$
$$T(0,0,1) = (0,0,1)$$

dim Ker T = 0 iff Ker T = $\{\vec{0}\}$, so no vector can have an image of (0,0,0).

Possible difficulty: T(1,2,3) = T(0,1,0) = (1,2,3), this also leads to Ker T $\neq \{\vec{0}\}$, since then T ((1,2,3)-(0,1,0)) = (0,0,0) from the linearity of the transformation.

2. dim Ker T = 1

$$T(1,2,3) = (1,2,3)$$
$$T(0,1,0) = (0,0,0)$$
$$T(0,0,1) = (0,0,1)$$

Possible difficulty: If two independent vectors have zero as their image, then the dimension will be two.

3. dim Ker T =2

$$T(1,2,3) = (1,2,3)$$
$$T(0,1,0) = (0,0,0)$$
$$T(0,0,1) = (0,0,0)$$

Advancing question: T(1,2,3)=(1,2,3) ; T(0,1,0) = T(0,0,1) = (1,1,1). What is the dimension of the kernel?

To solve this it is simplest to find dimension of the Image and use the dimension theorem, but it can also be found directly:

Ker T = $\{(x,y,z) \mid T(x,y,z) = (0,0,0)\}$ =

$= \{(x,y,z) \in \mathbb{R}^3 \mid T((x) \cdot (1,2,3) + (y-2x) \cdot (0,1,0) + (z-3x) \cdot (0,0,1)) = (0,0,0)\} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (x) \cdot T(1,2,3) + (y-2x) \cdot T(0,1,0) + (z-3x) \cdot T(0,0,1) = (0,0,0)\} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (x) \cdot (1,2,3) + (y-2x) \cdot (1,1,1) + (z-3x) \cdot (1,1,1) = (0,0,0)\} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (x+y-2x+z-3x, 2x+y-2x+z-3x, 3x+y-2x+z-3x) = (0,0,0)\} =$

$$= \left\{(x,y,z) \in \mathbb{R}\text{^}3 \mid \begin{cases} x + y - 2x + z - 3x = -4x + y + z = 0 \\ 2x + y - 2x + z - 3x = -3x + y + z = 0 \\ 3x + y - 2x + z - 3x = -2x + y + z = 0 \end{cases}\right\}$$

$= \{(x,y,z) \in \mathbb{R}^3 \mid x=0 ;\ y = -z\} = \{(0,y,-y) \mid y \in \mathbb{R}^3\}$

Thus, dim Ker T = 1, and this is not an example for dim Ker T =2.

4. dim Ker T =3

Since T(1,2,3) $\neq$ (0,0,0), then (1,2,3) is not in the kernel. Thus, Ker T $\neq \mathbb{R}^3$, Ker T is a subspace of $\mathbb{R}^3$ and so dim Ker T $\neq$ 3.

OR

dim Ker T = 3 iff dim Im T = 0, by the dimension theorem.

Im T = Sp{T(1,2,3), T(0,1,0), T(0,0,1)} = Sp{(1,2,3), T(0,1,0), T(0,0,1)},

so dim Im T $\geq$ 1, that is dim Im T $\neq$ 0, so for this T, dim Ker T $\neq$ 3.

Conclusion (5 minutes)

Linear transformations can be defined as a general element, on the elements of any basis or by the kernel or image. The connections between these various definitions will be stressed.

10.1.7 Diagonalization and Eigen Values (Week 13)

Lesson Goal: The question explores eigenvalues and diagonalization. This is the last topic of the course and utilizes all the previously learned topics. Calculating eigenvalues of a given matrix can be done ritually, by routines. This question asks for conditions on parameters for diagonalization. The routines are not sufficient.

The question also includes noting extreme cases, (e.g. $A=0$, A is a 1×1 matrix). These can be cases for which statements hold, and need to be taken into account. Students should be aware that these cases should be explored also.

Introduction: (7 minutes) Reminding the students of the basic definitions that they saw in class.

Definitions:

1. $\lambda \in F$ is an *eigenvalue* of a matrix $A \in F^{n\times n}$, if there exists $\vec{v} \in F^n$, $\vec{v} \neq \vec{0}$, such that $A\cdot\vec{v} = \lambda\cdot\vec{v}$.
2. The characteristic polynomial for matrix A is $p(\lambda) = |\lambda \cdot I - A|$, whose n roots are the n eigenvalues of A.
3. $AM(\lambda)$ =Arithmetic Multiplicity of λ = Multiplicity of λ as a root of the characteristic polynomial.
   The sum of the AM's is n.
4. $GM(\lambda)$ = Geometric Multiplicity = dimension of λ's Eigen space = $dim\{\vec{v} \mid A\cdot\vec{v} = \lambda\cdot\vec{v}\}$
5. $AM \geq GM \geq 1$
6. $\sum_{i=1}^{n} \lambda_i = tr(A)$ and $\prod_{i=1}^{n} \lambda_i = det(A)$
7. A is diagonalizable ⇔ for all eigenvalues AM=GM

Question: (15 minutes) The students will work in small study groups of 2-3 students in each group. The students will choose their own group.

Let A be an $n\times n$ complex matrix.

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ & \vdots & & \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

For what conditions on $a_1, a_2, \ldots, a_n$ is A not diagonalizable?

Possible Solution:

- If $a_1 = a_2 = \cdots = a_n = 0$ then A is diagonal, and thus diagonalizable.
- If $a_i \neq 0$ for some i, $1 \leq i \leq n$ (at least one a_i, the rest can be whatever) then $r(A) = 1$ and if $n > 1$ then A is not invertible, thus 0 is an eigenvalue with GM

$n - r(0I - A) = n - 1$. If $n = 1$ then A is a 1×1 matrix and is diagonal, thus diagonalizable.

- AM of 0 $\geq n-1 = \text{GM}$, by Theorem.
- Find value of λ_n, the last eigen value:

  If $\lambda_n = 0$ then $\text{AM}(0) = n \neq n-1 = \text{GM}(0)$ and A is not diagonalizable.

  If $\lambda_n \neq 0$ then $\text{AM}(0) = n-1 = \text{GM}(0)$ and then for λ_n it holds also that GM=AM, so A is diagonalizable.
- A is <u>not</u> diagonalizable when $a_i \neq 0$ for some i, $n \geq 2$ and $\lambda_n = 0$.
- By Theor em $\sum_{i=1}^{n} \lambda_i = tr(A)$, so $0 + \cdots + 0 + \lambda_n = a_1 + a_2 + \cdots a_n$
- A is not diagonalizable if $a_1 + a_2 + \cdots + a_n = 0$, for some i $a_i \neq 0$, $n \geq 2$.

General discussion (15 minutes) The students will be asked to present their solution on the board:

The connections between the different solution methods will be pointed out. The students can be asked if the solutions are the same and how do they connect.

Questions for further discussion (8 minutes)
These questions can also be used for students who need additional challenges during the small group period. These questions further explore the same concepts as above, while allowing the students to be more creative. If there is no time for these in class, they can be given as questions for the students to think about on their own.

1. Give a Linear Operator whose matrix representation in some basis is A.
2. Find Ker T, Im T, and the Eigen space of 0.

Solution:

Take n = 4 for an example, and $A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$:

Each column is Image of element in basis, so choose standard basis and:

$$T(1,0,0,0) = (0,0,0,0)$$
$$T(0,1,0,0) = (1,1,1,1)$$
$$T(0,0,1,0) = (-1,-1,-1,-1)$$
$$T(0,0,0,1) = (0,0,0,0)$$

Then $T(x,y,z,w) = (y-z, y-z, y-z, y-z)$, by properties of linear operator and basis.

$KerT = \{(x,y,z,w) \mid y - z = 0, x,y,z,w \in \mathbb{R}\} \Rightarrow KerT = \{(x,y,y,w) \mid x,y,w \in \mathbb{R}\}$

$ImT = \{(y-z, y-z, y-z, y-z) \mid y, z \in \mathbb{R}\} \Rightarrow ImT = \{(t,t,t,t) \mid t \in \mathbb{R}\}$

Eigen Space (0): Find Eigen Vectors:

$$(0I - A)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the System: $-y + z = 0$ yields 3 linear independent eigen vectors:

$$\{(1,0,0,0),(0,1,1,0),(0,0,0,1)\}$$

Eigen Space for 0 is the linear span of these vectors:
$Sp\{(1,0,0,0),(0,1,1,0),(0,0,0,1)\} = \{(x,y,y,w) \mid x,y,w \in \mathbb{R}\} = KerT$

Also note that $dimImT + dimKerT = 1 + 3 = 4$

Note: If $dimKerT = 3$ is T diagonalizable?
Not necessarily, since Ker T is the eigenspace of 0, the dimension is the GM. However, the AM can be different. For the above matrix the characteristic polynomial is: $p(\lambda) = \lambda^4$, the AM of 0 is 4, and A is not diagonalizable.

Conclusion (5 minutes) The connections between the kernel, the image and the Eigen space of 0 will be discussed.

10.2 Appendix B - Small Group Preliminary Analysis

	Group	Topic	Participants	Language	Interaction
1	S3 -1	Linear dependence	Three girls	English	The girls worked together, gave examples immediately, minimal discussion
2	S3 -2	Linear dependence	Alice & Ben	English	Ben tells Alice what to do, Much discussion
3	S4 -1	Linear Transformat ions	Sarah, Alice, Nicole	English	Sarah quiet. Others worked together, both suggest and discuss
4	S5 - 1	Eigen Values	Alice and 2 other girls	English	Alice tells others what to do, some discussion
5	S5-2	Eigen Values	Three girls	English	Discussion very not clear
6	W1-1	Complex Numbers	Three boys, Arabic	Arabic - Translated	Ahmed does most of talking at beginning, then involved others in discussion. Cadi suggests correct answer which was ignored initially.
7	W1-2	Complex Numbers	Segev, Ziv, Orr	Hebrew	All three discuss, made mistake and realized it. Attempted to build counter examples
8	W2-1	Matrices	Gal, Dor, Harel	Hebrew	Initially, Harel tells others what to do, but then makes an error. Then starts a discussion with all three involved
9	W3-1	SLE	2 boys Arab & Hebrew	Hebrew	Minimal discussion. When Ahmed suggests Yaniv does not listen.
10	W3-2	SLE	2 boys	Hebrew	Both state answers and claim “its simple/obvious”. No justifications given, so no discussion.
11	W3-3	SLE	2 boys	Hebrew	Leader and follower Basically, monologue of leader
12	W4-1	Linear Dependence	2 boys	Hebrew	Expert and follower Follower asks Eexpert for confirmation of math and social: “should we write it?”
13	W4-2	Linear Dependence	Yaniv & Hadar	Hebrew	Both discuss, both suggest. There is a mistake and then discuss
14	W4-3	Linear Dependence	Boy & girl	Hebrew	Leader & Follower Girl is expert

15	W5-1	Linear Transformations	Group 1 SLE	Hebrew	Work separately When minimal discuss: Ahmed treats Yaniv as expert
16	W5-2	Linear Transformations	Group 1 Matrix	Hebrew	Attempt to be an expert, but does not work. State answers with little justification or discussion.
17	W5-3	Linear Transformations	Boy & Girl	Hebrew	Girl is leader and tells Boy what to do
18	W6-1	Eigen Values	Yaniv & Nadav	Hebrew	Expert and follower
19	W6-2	Eigen Values	Gal, Dafna, Hadar	Hebrew	Two suggest things and all discuss things. Much discussion about why?
20	W6-3	Eigen Values	2 Boys and 1 Girl	Hebrew	Bar attempts to be expert, others don't accept him. Discussion about definition of diagonalizable

10.3 Appendix C - Hadar and Yaniv

10.3.1 Appendix C1 - Transcript of Hadar and Yaniv (Translated from Hebrew)

	Speaker	
26	Hadar	(Assertation) 2…a linearly dependent set, u belongs to V…
27	Yaniv	Yes. It (the assertation) is definitely true.
28	Hadar	A linearly dependent set, u belongs to V, all these together ($\{v_1, v_2, v_3, v_4\}$) are linearly dependent…Are you sure it's true?
29	Yaniv	If it ($\{u_1, u_2, u_3\}$) is already linearly dependent, and we add another vector, this subset is still linearly dependent
30	Hadar	Why? Take now 3 like this ($u_1 = (1,0,0,0)$)…take 3 like this [points to (1,0,0,0)], and now you add to them this ($u_4 = (0,1,0,0)$)…not necessarily (that the set is linearly dependent)"
31	Yaniv	What do you mean? What do you mean 3 like this?
32	Hadar	We said u_1, u_2, u_3 are linearly dependent.
33	Yaniv	No, but, here they (u_1, u_2, u_3) are linearly independent.
36	Hadar	Fine (downplaying). That's why I said let's take 3 that are dependent with u1 [looks at Yaniv]. Let's say here is 2, 3 and 4.
37	Yaniv	Nu. That's exactly what I am saying. If we add, doesn't matter what we add…these 3 vectors will still be dependent [looking at Hadar]
38	Hadar	The 3 (vectors) are (linearly dependent). But the fourth isn't. So, the entire set is linearly independent
39	Yaniv	Why?
40	Hadar	Because…Because it's possible. You can bring $u_1=(1,0,0,0)$; $u_2=(2,0,0,0)$; $u_3=(3,0,0,0)$; $u_4=(0,1,0,0)$.
41	Yaniv	Then it is still linearly dependent.
42	Hadar	How is it linearly dependent?!? [disagreeing]
43	Yaniv	No, it (the vector) isn't – but the set altogether is.
44	Hadar	Why? If you find scalars, that not all of them are zero…?
45	Yaniv	…that means that it is linearly dependent.
46	Hadar	And this (the linear combination) won't be equal to zero, because this (u4), you cannot neutralize if you don't put a zero for him

47	Yaniv	Yes. But it doesn't matter if he will be zero, if all the rest uh…if there is one …
48	Hadar	Then show me how
49	Yaniv	No, that's what I am saying. If there is at least one…uh…if there is one scalar
50	Hadar	Then show me how
51	Yaniv	No, that is what I am saying. If there is at least one…uh..if there is one scalar
52	Hadar	You are saying that if we cancel them out in a manner that is not zero and this one you brought for me that is zero
53	Yaniv	Exactly
53.1	Hadar	Ahhh…I understand
53.2	Yaniv	If there is one scalar at least that is different from zero...then it
54	Hadar	That's OK. So you are saying, like, that if in general there exists…exist at least two that are linearly dependent in the set, it doesn't matter which vector we add to them the set will still be linearly dependent.
55	Yaniv	Yes. Or…if it is 1 then simply
56	Hadar	I don't understand [to Yaniv]
57	Yaniv	Like, if there is…If one of these vectors is zero,
58	Hadar	Ummhmm [agreeing]
59	Yaniv	Then it doesn't matter by what we multiply it…there will still be zero.
60	Hadar	Ah! And that makes for us a linearly independent set.
61	Yaniv	No. A dependent set.
62	Hadar	Linearly dependent
63	Yaniv	Because there is one scalar that is different from zero.
64	Hadar	Hmm...then it is linearly independent.
65	Yaniv	Linear independence is what she wrote (looking at the board)
66	Hadar	There exist scalars that, that not all of them are zero
67	Yaniv	No, that…no
68	Hadar	Then if we found one scalar that is not zero
69	Yaniv	Yes?
70	Hadar	Then, if zero is in the set…then it (the set) is always linearly independent

71	Yaniv	No. Linearly dependence says that there are linear combinations
72	Hadar	That (linear dependence) means that alpha 1 is equal to alpha 2 is equal to zero, they are all equal to each other and they are equal to zero.
73	Y	That's not that it is linearly independent?
74	H	It's linearly independent
75	Y	Yes [both laugh]
76	Hadar	We are getting confused with the definition
77	Y	It's linear independent [laughing]
78	Hadar	Yes. Then…if we have all sorts in the set, and we put for all of them zero, but for the zero vector we put 3…
79	Yaniv	That's it. Exactly.
80	Hadar	Then the set becomes?
81	Yaniv	Linearly dependent.
82	Hadar	Dependent.
83	Yaniv	Yes.
84	Hadar	OK. That's the idea. The idea…exactly the conclusion at the end.
85	Yaniv	I hate these acronyms [laughs, hides mouth]
86	Hadar	Then wait a second…then..Wow! It's hard to realize this. That it () will always be dependent…We are saying it's always true [marks check on paper]
87	Yaniv	Yes. Its true. We need to prove it.
88	Hadar	Ummm
89	Yaniv	Ummm
90	Hadar	If this set is linearly dependent then it can be done, and this we can always multiply by zero
91	Yaniv	That's it. Let's assume that this set is linearly independent, and then
92	Hadar	Set? Ah! You want to assume by way of contradiction
93	Yaniv	Ah huh.(affirmative)
94	Hadar	We can do…if we said all these have an alpha 1, alpha 2
95	Yaniv	That's it. A subset.
96	Hadar	Alpha 3. And here an alpha 4….will be zero
97	Yaniv	No. Actually we do not have to assume by negation
98	Hadar	What is always, yes linearly dependent

99	Yaniv	We say that…there is here a subset that is linearly dependent
100	Hadar	That is, exist scalars such that the sum of this set will be equal to zero, Even if they (all the scalars) are not equal to zero.
101	Yaniv	Exactly
102	Hadar	And then if we add another vector, we can multiply it by zero
103	Yaniv	Then…uhhhh…we need to write it down?
104	Hadar	No. If you can repeat it orally on the board
105	Yaniv	On the board?! That's not orally (nervous laugh)
106	Hadar	Orally on the board (laughing)
107	Yaniv	Laughs
108	Hadar	Like, if someone wants to go to the board, OK OK. Let's continue.
109	Yaniv	OK
110	Hadar	No. We have the idea of the proof in our minds
111	Yaniv	Yes
112	Hadar	Let's continue.

10.3.3 Appendix C2 – Channels of communication analysis of Hadar and Yaniv

הדר	יניב	Non Verbal			Group 2Part1	1
Private		ה מקריאה את השאלה	דוגמא אחת נכונה ודוגמא אחת לא נכונה...	הדר	11:55	2
	Private	י כותב	אז...אה...נגיד, ננסה....4U	יניב		3
	Pro	ה על י, י מ על הדף	נניח אם ה...נניח אם המרחב הזה יהיה...אם V יהיה נניח וקטורים ל 4, ואלו יהיו....כן, וקטורים של 4	יניב		4
	Pro		ויש לנו רק 3 וקטורים.....אז הרביעי יכול להיות ת"ל	יניב		5
	Pro		ויכול להיות לא ת"ל	יניב		6
	Pro		נגיד אם זה הבסיסיים E1,e2,e3 וניקח 0 במקום 4 אז בת"ל	יניב		7
	Pro		משהו אחר אז ת"ל	יניב		8
	Pro		צריך להביא דוגמא ספציפית? או מספיק לכתוב אהה...	יניב	12:45	9
reac			דוגמא	הדר		10
	Pro		אולי פשוט אפשר לכתוב 2U+1U	יניב		11
reac			לא..זה אבסרקטי נראה לי...	הדר		12
pro			OK נניח ואז אתה אומר אחד ככה ואחד	הדר		13
private			זה ת"ל	הדר		14
pro			בשביל בת"ל צריך מספר אלא אם כן אני טועה, כאילו זה לא תאור טוב	הדר		15
pro			U1=(1,0,0,0); U2=(0,1,0,0); U3=(0,0,1,0); U4=(5,0,0,2)	הדר	14:05	16
pro			זה בתל	הדר		17
	reactive		כן	יניב		18
	pro		התת קבוצה הזאת עדיין ת"ל	יניב		19
react			למה?	הדר	14:42	20
pro			תיקח עכשיו 3 כאלה...	הדר		21
pro		על י	עכשיו אתה מוסיף להם את זה...לא הכרח	הדר		22
	reac		מה ז'תמרת? מה ז'תומרת 3 כאלה?	יניב		23
pro			אתה לוקח...נגיד...אמרנו ש 1U, 2U, 3U ת"ל אז נניח...יכול להיות שכאילו...	הדר		24
	re		לא.	יניב		25
	pro		אבל פה הם בת"ל	יניב		26
pro			הם יהיו פה צירוף ליניארי של ...	הדר		27
reac			לא, לא אמרו ש	הדר		28
	reac		לא. בדוגמא אמרו	יניב		29
reac			בסדר [ביטול] בגלל זה אמרתי שבא נקח 3 שהם תלויים לינארית עם 1U .	הדר		30
pro			בא נגיד שפה יש 2 3 ו4...	הדר		31
	reac		נו. זה בדיוק מה שאני אומר.	יניב		32
	pro		אם אנחנו נוסיף, לא משנה מה נוסיף...ה3 וקטורים האלה עדיין תלויים.	יניב		33

34		הדר	ה3 כן.			reac
35		הדר	אבל הרביעי לא.			pro
36		הדר	אז הקבוצה כולה תהיה בת"ל			pro
37	15:15	יניב	למה?		pro	
38		הדר	כי...כי אפשר			rea
39		הדר	אפשר להביא את U1=(1000); U2=(2000); U3=(3000); U4=(0100)			pro
40		יניב	אז זה עדיין תלוי לינארית		re	
41		הדר	איך תלוי לינארית?			pro
42		יניב	לא, הוא עצמו לא –		reac	
43		יניב	אבל כל הקבוצה ביחד כן		pro	
44		הדר	למה?			rea
45		הדר	אם תמצא סקלרים, שלא כולם אפס....			pro
46		יניב	...זה אומר שהיא תלויה לינארית		rea	
47		הדר	וזה לא יהיה שווה אפס, כי את זה, אתה לא יכול לאפס אם לא תשים לו אפס			pro
48		יניב	כן.		re	
49		יניב	אבל זה לא משנה אם הוא יהיה אפס, אם כל השאר אה... אם יש אחד..		pro	
50		הדר	אז תראה לי איך			pro
51		יניב	לא, זה מה שאני אומר. אם יש אחד לפחות..אה...אם יש סקלר אחד		rea	
52	16:00	הדר	אתה אומר שנבטל אותם בצורה שהיא לא אפס ואת זה הבאת לי שהיא כן אפס			reac
53		יניב	בדיוק		reac	
54		הדר	אהה...הבנתי			reac
55		יניב	אם יש סקלר אחד לפחות שהוא שונה מאפס..אז זה אההה		pro	
56		הדר	זה בסדר.			reac
57		הדר	אז אם בכללי קיימת 2 שהם תלויים לינארית בקבוצה, לא משנה מה נוסיף להם עדיין הקבוצה תהיה ת"ל			pro
58		יניב	כן..		rea	
59			או...אם זה 1 אז פשוט כי זה אפס		pro	
60		יניב	אם האפס בקבוצה אז היא אהה גם ת"ל		pro	
61		הדר	לא הבנתי			reac
62		יניב	כאילו, אם אחד מהוקטורים האלו הוא האפס		reac	
63		הדר	אמהם [הסכמה]			reac
64		יניב	אז לא משנה במה נכפיל אותו...עדיין יהיה אפס		pro	
65		הדר	אה.			rea
66		הדר	ואז זה גם עושה לנו קבוצה בת"ל			pro
67		יניב	לא.		reac	
68		יניב	קבוצה תלויה		pro	
69		הדר	תלויה לינארית.			reac
70		יניב	כי יש סקלר ששונה מאפס		reac	
71		הדר	הממה...אז היא בת"ל	על י מחייכת		reac
72		יניב	בת"ל זה מה שהיא כתבה	על הלוח	pro	

73		הדר	קיימים סקלרים ש...שלא כולם אפס.			reac
74		יניב	לא, ש... לא		reac	
75		הדר	אז אם מצאנו סקלר אחד שהוא לא אפס			pro
76		יניב	כן?		reac	
77	17:00	הדר	אז, אם אפס בקבוצה ...אז היא תמיד בת"ל			pro
78		יניב	לא.		reac	
79		יניב	ת"ל זה אומר שיש צירופים לינארים		pro	
80		הדר	זה אומר ש אלפא1 שווה לאלפא 2 שווה ל0			reac
81		הדר	כולם שווים אחד לשני והם שווים ל0			pro
82		יניב	זה לא שהיא בת"ל?		reac	
83		הדר	היא בת"ל			reac
84		יניב	כן	צוחקים	reac	
85		הדר	אנחנו מתבלבלים בהגדרה			reac
86		יניב	זה בת"ל	צוחקים	reac	
87		הדר	נכון			reac
88		הדר	אז אם יש לנו בקבוצה כל מיני סקלרים ונשים לכולם אפס, אבל לוקטור האפס נשים לו 3			pro
89		יניב	זהו. בדיוק.		reac	
90		הדר	אז הקבוצה הופכת להיות			pro
91		יניב	תלויה לינארית		rea	
92		הדר	תלויה			reac
93		יניב	כן	צוחק	reac	
94		הדר	OK זה הנקודה. זה הנקודה. ממש המסקנה בסוף	חיוך קטן		reac
95		יניב	אני שונא את ה???? האלו	צוחק ומסתיר פה	pro	
96		הדר	אז, רגע...אז... וואי קשה לתפוס את זה, שזה תמיד יהיה תלוי..			pro
97		הדר	אנחנו אומרים שזה נכון	וי על הדף		pro
98	17:43	יניב	כן. זה נכון.		reac	
99		יניב	צריך להוכיח את זה		pro	
100			שניהם בוהים בדף			
101	17:53	יניב	אה.....		private	
102		הדר	אם הקבוצה הזאת ת"ל אפשר לעשות את זה,			pro
103		הדר	ואת זה תמיד לכפול באפס			pro
104		יניב	זהו, נניח שהקבוצה הזאת בת"ל, ואז		private	
105		הדר	קבוצה...אהה! הבנתי. אתה רוצה להניח בשלילה			rea
106		יניב	אהההה		reac	
107		הדר	אפשר לעשות כל אלה			pro
108		הדר	אמרנו שיהיה להם אלפא 1 אלפא 2			pro
109		יניב	זהו. זה תת קבוצה		private	
110		הדר	אלפא 3. ופה האלפא תהיה 0			pro
111		יניב	לא. בעצם לא צריך להניח בשלילה		private	
112		הדר	כן ת"ל			pro
113		יניב	אנחנו אומרים ש...יש פה תת קבוצה שהיא תלויה לינארית		pro	
114	18:25	הדר	כלומר קיימים סקלרים כך שסכום הקבוצה הזאתי יהיה שווה לאפס	על הדף		reac
115		הדר	גם אם הם לא שווים לאפס	על י		pro

	reac	על הדף	בדיוק	יניב		116
pro		לדף	ואז גם אם נוסיף עוד וקטור אחר אפשר לכפול אותו באפס	הדר		117
	pro	לה	אז...אההה..צריך לכתוב את זה?	יניב		118
reac		לי	לא, אם **אתה** יכול לחזור על זה בע"פ על הלוח אז זה בסדר	הדר		119
	reac	לה	על הלוח?....אבל זה לא בעל פה על הלוח *מצחקק*	יניב		120
reac		לי	בעל פה על הלוח *מחייכת*	הדר		121
			שניהם צוחקים			122
			TA מדברת לכיתה			150

10.4 Appendix D – Transcript and Channels of communication analysis of Alice and Ben

1	Speaker	Mathematical Statement	Non verbal	Ben Channel	Alice Channel
2					
3	Alice		A moves to sit next to B's desk		Inter-personal
4	Ben	This is a 15 minute question. Are you ready?	Writing on paper	Private	
5	Alice	Starts reading question			Private
6	Ben	Wait a second … This could just be all zeros	Looking at paper only	Private	
7	Ben	(mumbles quietly, not clear)	Writes on paper	Private	
8	Alice	Reads Claim (above) Yes! Of course! Because it doesn't matter if	Looks at B		Inter Proactive
9	Ben		Looks only at paper	Private	
10	Alice	It means like always?	Looks at Ben		Inter Proactive
11	Ben	Is there a vector that you can add to this set will uhh…ummm..uhh..	Looking down	Private	
12	Ben	The question is - its a combination of these ummm vectors…ummm…wait a second.	Looking down	Private	
13	Ben	Umm It is a combination.	Looking down	Private	
14	Alice	I don't know if it's always true.	Sits up suddenly		Private
15	Ben	Yeah. It is true. It is true.	Matter of fact.	Inter Reactive	
16	Ben	If this is linearly dependent then this is linearly dependent	Looking down	Private	
17	Alice	But what if we add…			Inter Proactive
18	Ben	Forget it, it doesn't matter – its true		Inter Reactive	
19	Alice	Always?			Inter Proactive
20	Ben	Uhh.. so..		Inter Reactive	
21	Ben	If this is a linear dependent set, then there is a..		Private	

22	Ben	Then there is a…		Private	
23	Ben	One of these that is not zero,		Private	
24	Ben	then one of the alphas, one of the coefficients is not zero.		Private	
25	Ben	Like the coefficients are alpha, alpha times ; beta times		Private	
26	Alice	Yeah?			Inter Proactive
27	Ben	OK.	Nodding	Inter Reactive	
28	Alice	Signing attendance sheet			
29	Ben	U1, u2 u3, u4… Linear dependent, therefore ..		Private	
30	Ben	now we're going to have … ok… so option 1 : u… alpha1 is equal to ,		Private	
31	Ben	negative alpha u1 is equal to ummm…. beta u2 + gamma u3	Staring at ceiling	Private	
32	Ben	Now u1 is equal to… do we even need to do it this way?	looking at paper, talking to self)	Private	
33	Ben	I don't even know… fine… beta over alpha gamma over alpha….		Private	
34	Ben	so we have represented the u1 vector , in terms of a combination of the other vectors.	To Alice	Inter Proactive	
35	Ben	It shouldn't be 1 it should be three, I'll change the 3 to 1.	To self, correcting paper	Private	
36	Ben	It's usually the last one.	To self	Private	
37	Ben	This basically means that alpha is not equal 0. That's all that means.	To Alice	Inter Proactive	
38	Ben	I don't know. I'm saying that they're linear dependent.	To paper	Private	
39	Ben	We're saying that alpha u1 beta u2 gamma u3 umm…,	To self, writing on paper	Private	
40	Ben	where alpha beta gamma are not all zero… equal to zero		Private	
41	Ben	So if one of them exist , we can divide by it.	Stares at Alice until she nods	Inter Proactive	
42	Alice	umm hmmm	nodding		Inter

					Reactive
43	Ben	We can work with it, this one here will be a combination uhh…so…	To paper	Private	
44	Ben	if you add another vector….	To paper	Private	
45	Ben	if you add u4… then…we're going to have Alpha u1 beta u2 gamma u3 theta u4 …	To paper	Private	
46	Ben	now we want that to be equal to the zero vector…	To paper	Private	
47	Ben	in order to check whether or not they're dependent or independent.	To paper	Private	
48	Ben	And we know that alpha is not equal to zero…. umm… wait a second… wait a second…	To papere	Private	
49	Ben	What do you think?	(To Alice)	Inter Proactive	
50	Alice	Yeah, it works			Inter Reactive
51	Alice	because if you subtract, right? subtract, minus all of this stuff and divide by alpha then you get, you get …			Inter Reactive
52	Ben	negative beta over alpha u2 minus gamma over alpha u1,	Starts writing and talking:	Private	
53	Ben	instead of 3 cause…,	To paper	Private	
54	Ben	minus theta over alpha u4 equals v3, therefore …ummm…	To paper	Private	
55	Ben	Kay, this is not the proof at all, this is just like the idea of the whole thing	To paper	Inter Proactive	
56	Alice	Why is this ????? I mean …			Inter Proactive
57	Ben	We need to write alpha and beta		Inter Reactive	
58	Ben	Beta exists in all different Umm … ok, and also…		Inter Reactive	
59	Ben	You write it neatly. (the proof is finished)		Inter Proactive	
60	Alice	You want me to write it out? Thank you.	(laughing)		Inter Reactive
61	Ben	Yeah – you can do that.		Inter Reactive	
62	Ben	And show that you, we know this		Inter Proactive	

63	Alice	Ok. Fine, fine			Inter Reactive
64	Ben	We have the idea		Inter Proactive	
65	Alice	Ok. We have alpha 1, keep track of it – yeah?			Inter Proactive
66	Alice	Option 1… Wait - why option 1?			Inter Proactive
67	Ben	Well… Because we don't know what vector…		Inter Reactive	
68	Ben	we don't know which alpha, beta or gamma is equal, not equal to zero.		Inter Reactive	
69	Alice	You can choose whichever you want because…			Inter Proactive
70	Ben	Technically we have to write it … umm…		Inter Reactive	
71	Alice	Yeah. But why? (laughing)			Inter Proactive
72	Ben	That (mumble) was important		Inter Reactive	
73	Alice		Laughing		
74	Ben	Cool. Then we're going to say that…		Private	
75	Ben	This is the definition of independence, of an independent set- right?		Inter Proactive	
76	Alice	Yeah. You got it. Yeah That's, that's …			Inter Reactive
77	Ben	This is saying that's true		Inter Reactive	
78	Alice	Yes, it is.			Inter Reactive
79	Ben	Then if you add another one to that second one, you can still divide by the alpha and then you will have a combination…		Inter Reactive	
80	Ben	Cause this is not equal to zero.		Inter Reactive	
81	Alice	But			Inter Proactive
82	Ben	But if beta is equal to zero, it doesn't make a difference		Inter Reactive	
83	Ben	it's still linear dependent		Inter Reactive	

84	Alice	But if beta equals zero then <mumble>			Inter Proactive
85	Ben	Yeah, you can still make it zero		Inter Reactive	
86	Alice	Wait! Let me just make sure I understood, 'cause now I'm … One side satisfies - Alpha times …Zero… fine …			Inter Proactive
87	Alice	Plus, I will take… Then we'll take the one that has a coefficient that is not zero…			Inter Proactive
88	Ben		Playing with pen but not talking		
89	Alice	Well take vector u1 that has a coefficient that is not zero - ok? So we take			Inter Proactive
90	Ben	Linear		Inter Reactive	
91	Alice	We can add another vector to the set			Inter Proactive
92	Alice	and then we're going to say umm Alpha plus… yeah, alpha beta gamma			Inter Proactive
93	Ben	As long as it's not zero, we can still bring it over to the other side, and divide.		Inter Reactive	
94	Ben	Which gives … which is a linear combination … is		Inter Reactive	
95	Alice	Wait. wait. There's one more, one more step.			Inter Proactive
96	Alice	You subtracted this.			Inter Proactive
97	Ben	Here's the zero		Inter Reactive	
98	Alice	How does this show? we need to show			Inter Proactive
99	Ben	This shows that these are a combination of the previous ones		Inter Reactive	
100	Alice	Yeah. but why does that matter?			Inter Proactive

101	Ben	Because a dependent, linearly dependent set is a linear combination of, has a vector that is a combination of the other vectors.		Inter Reactive	
102	Alice	Yeah. But why?			Inter Proactive
103	Ben	What do you mean – why?		Inter Proactive	
104	Alice	Why? why does that come from the definition of …alpha,			Inter Proactive
105	Ben	It does		Inter Reactive	
106	Alice	Yeah, but why?			Inter Proactive
107	Alice	If we subtract u3 from here			Inter Proactive
108	Ben	It wouldn't be zero		Inter Reactive	
109	Alice	Then you have 0, the zero vector.			Inter Proactive
110	Alice	so then you have at least you have non-zero coefficients			Inter Proactive
111	Ben	I don't think that's the reason.		Inter Reactive	
112	Ben	I think the reason is because v3 is a combination of the other vectors,		Inter Proactive	
113	Ben	therefore is dependent		Inter Proactive	
114	Ben	and it doesn't need, it doesn't need to be included in the set… in order for it to be… uhh…spanning… uhh, spanning all the numbers		Inter Proactive	
115	Alice	I agree with you, but I don't understand why…			Inter Proactive
116	Ben	I don't know why the theorem works…		Inter Reactive	
117	Alice	Why?	Laughing		Inter Proactive
118	Ben	I don't know – ok?		Inter Reactive	
119	Alice	Ok we're moving along			Inter Proactive
120	Ben	You ask her why, not me.		Inter Reactive	
121	Alice	Ok. Fine. Fine.			Inter

					Reactive
122	Ben	I’ll write it and if you have an issue - I’ll figure it out		Inter Proactive	
123	Alice	Ok.	Laughing		
124	Ben		(writing down)		
125	Alice	Can I ask you a question? (To TA)			

עידוד השתתפות חקירתית באלגברה לינארית דרך הוראה עשירה בשיח - החשיבות של למידה ברמת האובייקט ולמידה ברמת העל

מרים נחמה ולך

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חיבור על מחקר
לשם מילוי חלקי של הדרישות לקבלת התואר דוקטור לפילוסופיה

מרים נחמה ולך

הוגש לסנט הטכניון - מכון טכנולוגי לישראל

אלול תשפ"ב, חיפה, ספטמבר 2022

המחקר נעשה בהנחייתם של
פרופ"ח עינת הד-מצויינים מהפקולטה לחינוך למדע וטכנולוגיה
ופרופ"ח רם בנד מהפקולטה למתמטיקה.

אני מודה לטכניון – המכון הטכנולוגי לישראל על תמיכתו הנדיבה בהשתלמותי.

אני מודה למר ליפה משורר וגברת יהודית משורר, תושבי כפר שמריהו, ישראל, על תמיכתם בשנה"ל תש"פ במלגת ליפה ויהודית משורר.

אני מודה לגברת ג'סיקה אלין, תושבת מדפורד, אוראגון, ארה"ב, על תמיכתה בשנה"ל תשפ"א במלגת עמנואל גוטסמן.

תקציר מורחב

אנשי כנסת הגדולה, שתיקנו את נוסח התפילות, כללו בתפילה הנאמרת בכל יום את המילים: "תן בליבנו בינה להבין ולהשכיל לשמוע ללמוד וללמד לשמור ולעשות" (ברכות קריאת שמע). המפרשים מסבירים שצמד הפעלים ללמוד, ללמד, לשמור ולעשות מלמד אותנו שדרושה השתתפות פעילה להוראה פורה ולמידה משמעותית. בנוסף, בספר משלי כתוב "חֲנֹךְ לַנַּעַר עַל פִּי דַרְכּוֹ" (פרק כ"ב פסוק ו), ומאיץ בנו לבחור בדרכי הוראה שלוקחים בחשבון את צורכי הלומד. הוראה מסוג זה נקראת בספרות המודרנית הוראה פעילה ממוקדת לומד, וזוכה לשבחים ולעידוד על ידי חוקרים ואנשי מקצוע.

במתמטיקה, הוראה ממוקדת לומד תומכת במעורבות משמעותית של הלומדים במתמטיקה, בשיתוף פעולה בין הלומדים לקראת הבנה ובשילוב שיטות הוראה שוויוניות. מחקרים שונים המתארים הטמעה של שיטות הוראה ממוקדות לומד במסגרות אוניברסיטאיות, מראים כי הצלחת הלומדים, כפי שהיא נמדדת בהישגים, נתמכת על ידי שיטות אלו. אולם, תהליך הלמידה, ולא רק תוצאת הלמידה, חשוב. יחד עם זאת, בעוד שמחקרים הצביעו על תרומת שיטות מבוססות לומד להישגים במתמטיקה , ישנם מחקרים המצביעים על חסרונות להוראת מתמטיקה מבוססת חקירה ודיונים. בין היתר, הלמידה בקבוצות קטנות, ללא מומחה, עשויה להרחיק את הלומדים ממטרות תוכנית הלימודים. בנוסף, יתכנו הסחות דעת בשל אינטראקציות חברתיות או תקשורת לקויה בין חברי הקבוצות. לפיכך, למחקר זה היו שתי מטרות עיקריות. הראשונה הייתה להתאים את שיטות ההוראה שהוכחו כמקדמות למידה חקירתית עשירה בשיח לקורס אלגברה לינארית באוניברסיטה, על מנת לתמוך בהשתתפות ולמידה חקירתית של לומדים ולעודד אותן. השנייה הייתה לבחון היבטים שונים של הוראה ממוקדת לומד ואת התהליכים הכרוכים בה כדי להבין טוב יותר מה תומך בלמידה ומה מעכב אותה במסגרות לימודיות מסוג זה.

המסגרת התיאורטית ששימשה למחקר הייתה המסגרת הסוציו-תרבותית הקומוגניטיבית. מסגרת זו נבחרה כיוון שהיא מאפשרת תיאור וניתוח של תהליכי למידה מתמטיים, כמו גם התייחסות הוליסטית לתכנים ולאינטראקציות חברתיות בהוראה ולמידה. קומוגניציה מגדירה את הלמידה כהצטרפות לקהילה וכשינוי בשיח של הלומד תוך כדי ההשתתפות. על פי הגישה הקומוגניטיבית, משתתפים חדשים מחקים תחילה את המומחים באופן ריטואלי, ולאחר שהפנימו את חוקי הקהילה, פועלים באופן עצמאי וחקירתי. השתתפות ריטואלית מאופיינת בתמרון סמלים מתמטיים הממוקדים בהליך. לעומת זאת, השתתפות חקירתית מאופיינת בהליכים אוטונומיים ויצירתיים הממוקדים בתוצאת ההליך. כך, לימוד מתמטיקה הוא תהליך הדרגתי של דה-ריטואליזציה. תהליך הוראה-למידה מוצלח הוא כזה התומך באופן מירבי במעבר של הסטודנטים מיישום רוטינות ריטואליות ליישום רוטינות חקירתיות.

למידה, על פי התיאוריה הקומוגניטיבית, מתקדמת בשתי רמות: רמת האובייקט ורמת העל. בלמידה ברמת האובייקט הלומדים מרחיבים את מערך ההיגדים שהם מאמצים על עצמים המוכרים להם. בלמידה ברמת-העל הלומדים נחשפים לכללים חדשים ומשנים את ההליכים שעל פיהם הם מאמצים היגדים בנוגע לעצמים אלו. לדוגמא, עבור לומד שאימץ את הכלל לחיבור מספרים שלמים תידרש למידה ברמת העל כדי ללמוד את הכלל של חיבור שברים. באלגברה לינארית ברמה אוניברסיטאית, למשל, השקילות של ייצוגים השונים (למשל, קבוצות וקטורים, פתרון של מערכות משוואות ליניאריות, מטריצה וכו') מהווה אתגר משמעותי ללומדים, ודורשת למידה ברמת-על. כחלק מלמידה ברמת על יש אימוץ של היגדים בשיח חדש, דבר הדורש תהליך של עיצום. עיצום הוא תהליך בו הלומד מחליף תקשורת על תהליכים עם תקשורת על תוצאת התהליך ובכך בונה עצם מתמטי דיסקורסיבי. לדוגמא, האמירה "יש ארבע עוגיות" היא עיצום של הסיפור התהליכי "כשאני שר את הספרות ומצביע על כל עוגיה בנפרד, השיר מסתיים בארבע". תהליך העיצום מייתר את התהליך של פעולות על העצם ואת הסובייקט הפועל, וכך ניתן לתקשר על העצמים המתמטיים ביתר יעילות.

במסגרת מחקר זה פיתחתי סדנאות ששולבו בקורסי אלגברה לינארית כדי לעודד ולתמוך בהשתתפות חקירתית בשיח של אלגברה לינארית . סדנאות אלו השתמשו בשיטות הוראה מבוססות דיון, וכללו דיונים סביב משימה נתונה בקבוצות קטנות ובמליאה.. הסדנאות התקיימו במהלך שנות הלימודים תשע"ט ותש"פ כהעשרה לסטודנטים בסמסטר הראשון שלהם בלימודי הנדסה, מדעי המחשב או מתמטיקה. חלק מהסדנאות התקיימו באנגלית (לסטודנטים הלומדים במסגרת בינלאומית) והחלק מהסדנאות התקיימו בעברית בקבוצות של בין 10-60 סטודנטים. כל סדנה הוקלטה באמצעות מספר מצלמות נייחות.

באמצעות המסגרת הקומוגניטיבית, המחקר בחן היבטים שונים של תהליכי למידה הכרוכים בסדנאות. ראשית, נבדק הפוטנציאל של משימות הסדנא לתמיכה בהשתתפות חקירתית ולעידוד למידה. לאחר מכן, נבדקו ההזדמנויות ללמידה חקירתית שניתנו בפועל לסטודנטים בדיונים בכיתה. לבסוף, נבחנו תהליכי הלמידה בדיונים מתמטיים שבוצעו בקבוצות קטנות של סטודנטים.

על בסיס התיאוריה הקומוגניטיבית פיתחתי כלי – Discourse Mapping Tree (DMT) – למיפוי העצמים המתמטיים הנכללים במשימות.כלי זה נבנה על סמך ניתוח תת-שיחי האלגברה הלינארית שהמשימה מזמנת. תת-שיחים אלו כוללים, למשל, תת-שיחים של פונקציות, של מטריצות ושל מרחבים וקטוריים. כלי ה-DMT אפשר לי: א) לבחון אילו היגדים על אובייקטים מתמטיים הסטודנטים יכולים לחבר, בהתבסס על המשימות שפותחו, ו-ב) לקבוע אם למשימות יש את היכולת לעורר דיונים, לאלץ סטודנטים לחבר מימושים שונים והיגדים המקשרים ביניהם, ולספק למורה הזדמנויות להדגשת קישורים פחות מוכרים. הרחבה של כלי זה – Discussion Discourse Mapping Tree (DDMT), – שימשה למיפוי היישום של המשימה בפועל בדיונים בכיתה. תהליכי הלמידה בקבוצות קטנות נבחנו באמצעות ניתוח של המתמטיקה וערוצי התקשורת בלמידת עמיתים של סטודנטים.

ניתוח המשימות הראה שלמשימות יש פוטנציאל הן ללמידה ברמת האובייקט והן ללמידה ברמת העל. הפוטנציאל ללמידה ברמת העל המוטמעת במשימות כוללת היכרות עם ותרגול של כללי על הקשורים לעצם המתמטי שבמשימה. ניתוח זה גם תמך בגיבוש הגדרה אופרטיבית של פוטנציאל של משימות לתמוך בלמידה חקירתית ובחינת מאפיינים ספציפיים של משימות אלה, כגון הכללה של איפיון תהליך פתרון ככזו שכוללת בתוכה מבוי סתום המאלצת את הלומד לחפש דרך אחרת לפתרון.

המשימות המעוצבות העניקו הזדמנויות הן ללמידה ברמת האובייקט והן ללמידה ברמת העל, והסדנאות המיושמות נבחנו להערכת המידה בה ניצלו הסטודנטים את ההזדמנויות ללמידה ברמת העל. ברוב המקרים, הדיונים הכיתתיים כללו הזדמנויות רבות, שנתמכו על ידי המורה, ליצירת היגדים בשיח החדש התומך בלמידה ברמת על. הניתוח באמצעות ה-DDMT הדגים היבטים שונים של השתתפות הסטודנטים בדיונים מתמטיים. ראשית, נמצא כי לעתים קרובות היה תת-שיח דומיננטי שהסטודנטים נאחזו בו, ומהמורה נדרש לעודד באופן אקטיבי שימוש של הסטודנטים בתתי-שיח אחרים. שנית, בניית הקשרים בין תתי-שיח הייתה תלויה בהיכרותם של הסטודנטים עם הנרטיבים והאובייקטים שבתת-השיחים. לבסוף, לקישורים שיזמה המורה היה תפקיד מכריע בחיבור מימושים שונים בתתי-שיח מרובים ובתמיכה בבניית קישורים בין תת-השיחים.

הדיונים בכיתה, בהנחיית המורה, כללו הזדמנויות רבות ללמידה ברמת העל. אולם, עיקר החקירה וההתמודדות העצמאית של הסטודנטים עם המשימות התרחשו בשלב הלמידה השיתופית בקבוצות קטנות. בחנתי לעומק שתי אפיזודות למידה של זוגות סטודנטים בלמידה שיתופית ללא תמיכה על ידי מומחה. לזוג אחד הייתה אינטראקציה שוויונית בעיקרה, שבה התקשורת הייתה בעיקר בערוץ הבין-אישי. התקשורת בערוץ זה תמכה בתקשורת יעילה בין הסטודנטים. למרות זאת, אחד הלומדים לא שינה את הרוטינות הלא נכונות שלו. ניתוח של השיחים של הסטודנטים הראה שוני ברמת העיצום. סטודנט אחד התקדם בתהליך העיצום, והוא ניצל את ההזדמנויות שהדיון העניק לו לקידום הלמידה. בניגוד אליו, חברתו לקבוצה, שהייתה רק בתחילת תהליך העיצום, לא הצליחה להשתתף בשיח החדש. היא אמנם ייצרה נרטיבים חדשים ברמת האובייקט, אך לא הצליחה להתקדם ברמת העל.

זוג נוסף שנבדק לעומק הראה אינטראקציה לא שוויונית. באותה אינטראקציה התקשורת התקיימה בערוצים שונים. נמצא כי סטודנט אחד לקח על עצמו את תפקיד המנהיג והמומחה, ואילו לסטודנטית שהשתתפה אתו בדיון לא ניתנה הזדמנות להתעמת עם הצהרותיו המתמטיות או לבחון את נכונותם. ההיגדים המתמטיים הלא נכונים של הסטודנט התבססו על רמת עיצום לא מתקדמת דיה, ואילו הסטודנטית השותפה לא העלתה ביקורת על ההיגדים. הסטודנטית התאימה את עצמה לתפקיד שניתן לה כעוקבת, ולא ניתנה לה אפשרות לבטא את רעיונותיה. במקרה זה, הן הלמידה ברמת האובייקט והן הלמידה ברמת העל נפגעו.

המסקנה העולה ממכלול הניתוח - ניתוח ה-DMT של המשימות, ניתוח ה-DDMT של כלל הדיונים בכיתה והניתוח של שני זוגות הסטודנטים - היא שהמשימות שפותחו יצרו הזדמנויות הן ללמידה ברמת האובייקט והן ללמידה ברמת העל, במסגרת דיונים במליאה בתמיכתה של המורה. עם זאת, הדיונים בקבוצות קטנות תמכו בעיקר בלמידה ברמת האובייקט, וגם זאת, רק כאשר התקשורת בין הסטודנטים היתה שוויונית והשלב בו נמצאו הסטודנטים בתהליך העיצום אפשר זאת. כלומר, מחקר זה הראה שלמידה מבוססת דיון ולמידה שיתופית

במתמטיקה ברמה אוניברסיטאית יכולה להיות פורה, כאשר לוקחים בחשבון את המשימות, תפקיד המורה, האינטראקציות בין חברי הקבוצה וסוג הלמידה הנדרשת.

למחקר זה תרומה מעשית, מתודולוגית ואמפירית. מעשית, מערכי השיעור והמטלות יכולים לשמש מורים אחרים, ובמינוף התובנות מפרויקט זה ניתן לפתח משימות דומות לנושאים אחרים במתמטיקה ברמה אוניברסיטאית. מתודולוגית, המחקר הנוכחי פיתח כלי לבחינת פוטנציאל של משימות ברמה אוניברסיטאית, ה-DMT, וכלי לבחינת יישום של המשימות, ה-DDMT. כלים אלו יכולים לשמש לבחינה של נושאים נוספים ורמות לימוד מגוונות. לבסוף, מחקר זה הראה את החשיבות של ההבדל בין למידה ברמת האובייקט ולמידה ברמת על. עולה ממנו כי שיח עמיתים יכול להיות מוצלח ללמידה ברמת האובייקט. עם זאת, למידה ברמת על דורשת תמיכה של מומחה המכוון לכללים המרומזים שהסטודנטים צריכים ללמוד, ובעיקר לחיבור בין תת-השיחים השונים. לפיכך, עיצוב שיעור בהוראה ממוקדת לומד במתמטיקה ברמה אוניברסיטאית צריך להיות מותאם להבדל בין למידה ברמת האובייקט ללמידה ברמת העל, ושיטות ההוראה צריכות להיות מותאמות לסוג הלמידה הנדרש.