mapping objectification in early algebraic discourse

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This study is part of a larger study designed to produce tools for mapping students' arithmetic and algebraic discourse based on the commognitive theory. Commognition theorizes learning as a transition from ritual to explorative participation in the mathematical discourse, characterized by increased objectification. We propose a method that examines the level of objectification in 7-8 grade students’ algebraic discourse, while identifying the objects about which the narratives in the students' discourse revolve. We demonstrate this method on one algebraic problem. Our findings show that our method succeeds in differentiating canonical as well as non-canonical solutions and in characterizing them on a continuum towards the objectification of a generalized number, which can be realized as a variable or an unknown.

Algebraic thinking is a critical milestone in mathematics learning. Successfully learning algebra in school opens the gate to the rich world of academic mathematics. Yet, the transition students experience in school from arithmetic to algebra is accompanied with some challenges that are not yet fully addressed by the research community (Kieran, 2022). Some of the efforts to overcome these challenges led mainstream studies to develop diagnostic tools that reflect snapshots of students’ performance at particular points in time (e.g., Stacey et al., 2018; Klingbeil et al., 2024). These tools mainly focus on student’s procedural challenges and misconceptions, yet they do not offer much insight into the developmental trajectory of algebra learning. Indeed, there is need for assessment tools that rely on a developmental theory of algebra learning, including the ways in which algebra learning builds on more basic, arithmetic, mathematical knowledge (Warren et al., 2016; Kieran, 2022). One such theory has been offered by Caspi and Sfard (2012), who suggested that algebra develops as a meta-discourse on arithmetic. So far, the commognitive theory has been useful for providing diagnostic tools for mapping students arithmetic discourse (Ben-Yehuda, 2003; Heyd-Metzuyanim et al., 2022). However, applying these tools to map students’ algebraic discourse has remained a challenge, since a main part of the theory of de-ritualization, concerning objectification, has not been operationalized in terms of mastery of algebraic skills. Our aim is to address this gap, by offering a methodological tool that discerns different stages of objectification in the algebraic discourse.

Theoretical framework

The purpose of learning according to the commognitive paradigm (Sfard, 2008) is the learner's transition from ritual participation in discourse—focused on performing procedures that imitate the performance of an expert—to explorative participation—focused on producing narratives about mathematical objects in discourse. A central aspect of this process is the objectification that occurs when the learner moves from using keywords or signs as “empty” signifiers (that do not denote anything but themselves) to those that denote objects (Lavie et al., 2019). The process of algebraic object formation germinates when students make their first steps in meta-arithmetic discourse (Caspi & Sfard 2012) and continues when students use symbols that signify algebraic objects. These will be called symbolically mediated objects e.g., $\frac{x}{7}$ that signifies a number divided by seven. Another aspect of explorative participation concerns the characterization of narratives in discourse. In explorative participation, the learner attempts to produce stories about mathematical objects in a consistent and logically connected manner (Baccaglini-Frank, 2021).

This study is part of a large-scale study designed to produce tools for mapping students' arithmetic discourse and algebraic discourse. In previous parts of the study, a tool was developed for mapping elementary school graduates' arithmetic discourse on the continuum of ritual and exploration (Heyd-Metzuyanim et al., 2022), as well as a tool for visually mapping aspects of objectification in algebraic discourse (Shahla Demirdjian, 2025). However, objectification in algebraic discourse remains difficult to map due to several challenges. First, there are several objects in algebraic discourse (variable, unknown, equation, etc.). Second, previous studies (Cohen, 2024; Shahla Demirdjian, 2025) have indicated a great deal of variation in the ways in which seventh- and eighth-grade students solve algebraic problems noncanonically, which has made it difficult to map the development of discourse along a single axis. This mapping is important for progress toward quantification, which will allow the application of commognitive theory to larger samples.

At the beginning of learning algebra, the key change required in students’ discourse is a shift from arithmetic discourse, which deals with specific numbers, to discourse that deals with generalized (non-specific) numbers. Therefore, to characterize the degree of objectification in algebraic discourse, we ask: *To what extent do students’ implicit and explicit stories in algebraic discourse revolve around generalized numbers?*

Research Methodology

The data are taken from a series of EADP (Early Algebraic Discourse Profile) interviews with 10 seventh grade students from different schools in Israel, and at different achievement levels. The EADP interview was conducted in a "think-aloud" format and contains 13 algebraic tasks taken from the seventh grade Ministry of Education exams and from Caspi (2014). To demonstrate the analysis method we developed, we focus on one of the questions - the "I thought of a number" problem:

*I thought of a specific number. If I multiply it by seven and subtract from the product fifty-four, I will get the number I was thinking of. What is the number I was thinking of? Explain how you solved it.*

We should emphasize that the problem is communicated within the meta-arithmetic discourse and is telling a story about a generalized unknown number (“I thought of a specific number” and “the number I was thinking of”). The interviews were video recorded using two cameras, one recording the writing and the other the student's face and were fully transcribed. To demonstrate the differential sensitivity of the analytical method, five students were selected (Liat, Gil, Tom, Mika and Alon – pseudonyms) but each of them used a different procedure and not all of which were canonical in the algebraic discourse. The analysis includes two stages, the first, detecting narratives in the students’ discourse, and the second, detecting the subject of each narrative, deciding whether it is a signifier of a mathematical object and if so, is it a generalised number. The narratives are sorted into three categories according to the mathematical object they are about: specific numbers, verbally generalized numbers, and symbolically mediated generalized numbers. Each narrative is marked as explicit if it was talked about directly or implicit if it was only implied by the student’s descriptions of his or her procedures.

Findings

The analysis focuses on the solutions of five students, four of which produced non-canonical algebraic procedures: **Mika** focused solely on a trial-and-error procedure and found that the solution is the number nine. **Liat** declared at the beginning of the solution that she could solve the problem “in two ways”. She then tried to construct an algebraic expression but was unable to create an equation. She tried using arithmetic calculations, but this procedure was also unsuccessful, and she gave up. **Gil** also began solving the problem algebraically and, like Liat, encountered difficulty in creating an equation. However, when he reached a dead end with the equation, he switched to a trial-and-error procedure and like Mika, found that the number was nine. **Tom** began by declaring that he had no idea how to solve the problem, but with the interviewer’s encouragement, he created an equation. However, the equation was not canonical and the operations on it were not canonical, so he was unable to reach a solution. Finally, **Alon**, is the fifth student and the only one that formed an equation and solved it canonically.

Below is an analysis of the the five students’ solutions according to the explicitness of their narratives and the objects they revolve around.

In Mika’s discourse we found mostly narratives about specific numbers, yet we also found an implicit narrative about a generalize number. Mika’s solution is based solely on performing procedures on specific numbers. She starts her solution with:

237 Mika: …say, well, I think about, say, (the number) two, I multiply it by seven, no, it doesn’t make sense, because it doesn’t get the…say I think of (the number) ten…no, this also doesn’t make sense, because if I think of ten… If I multiply by seven it equals seventy but, like, fifty-four minus seventy I already know by heart it cannot be two (*corrects to “ten” after interviewer asks, “why two?”)*.

Mika clearly performs a procedure of trial and error by substituting the sequence of whole numbers: two and ten. Later she tries five and then nine, which leads her to concluding the number is nine.

Although Mika’s procedures involve only specific numbers, her usage of a series of these numbers implies that she interprets “the number I thought of” appearing in the problem’s text, as any number, and therefore, a generalized number.

In Liat's discourse, no narratives were found about generalized numbers, but only about specific numbers. For example, in the following passage:

272 Liat: Umm... x, I can do (it in) two ways, x times seven equals... uh... to...something,

In this excerpt, although Liat writes a symbolic expression ($x∙7=$), the expression does not tell a story about mathematical objects. It talks only about actions that Liat can do. After the symbolically mediated attempt does not lead Liat to a satisfactory description of the situation, Liat moves, without realizing it, to trying to solve a problem similar to the given problem in which the resulting number is not "the number I was thinking of" (a generalized number), but the number seven. Such a problem would be indeed solvable using the reverse arithmetic operations. Liat performs on these reverse operations described, calculating the expression $\frac{7+54}{7}$ and then $\frac{61}{7}$. During this attempt, Liat's narratives revolve exclusively around specific numbers. After encountering difficulties with this attempt as well, she starts the procedure over again. During this process, she writes the expression $7∙x-54$. Regarding this procedure, Liat notes that "x is the specific number." This narrative ostensibly revolves around a generalized number, but it contains nothing more than a matching of signs (“x” to “the specific number” as worded in the task). Therefore, in Liat's discourse, there are no narratives whose object is a generalized number.

In Gil's discourse, one can find narratives that are mostly concerned with specific numbers. However, we can also find in his procedures implicit and explicit narratives about generalized numbers. In the following we explicate the different kinds of narratives we found in his discourse.

190 Gil: Okay, fine. So I'll do it x plus fifty four divided by seven.

Similarly to Liat’s narrative [272], Gil’s narrative in turn 190 describes a symbolic expression $\left(\frac{x+54}{7}=\right) $as something Gil does and not as a narrative about a generalized number. Gil also performs a trial-and-error procedure and finds the desired number. He says: “the number is nine” [194]. This utterance is indeed a narrative about a generalized number, as the signifier “the number” signifies the “the number I thought of” in the given problem, and the story told here regarding that number is that it is nine.

When depicting the procedure that yielded the canonical narrative Gil elaborates:

207 Gil: I tried to put in a number that was like a two-digit number and then I realized that it didn't make sense.

209 Gil: And then I tried... I added thirteen and then I realized that it didn't make sense as if it was too big a number, I tried the largest single digit, and I just got nine, I got nine.

Similarly to Mika, the fact Gil considers “a number” as a signifier of “any number” and his substitution procedure imply that the narrative given in the problem is about a generalized number. In addition, in this explanation, there are two verbal descriptions of generalized numbers: "a two-digit number" and "a single-digit number" which Gil tells the following two narratives about: When a two-digit number is substituted for “the number I thought of” it yields a number which is too big, and - the largest one-digit number is nine. To conclude, we found in Gil’s discourse symbolic signifiers detached from any object, narratives around specific numbers and implicit and explicit narratives (verbal, non-symbolic) around generalized numbers.

In Tom's discourse, one can find symbolically mediated narratives about generalized numbers.

 21 Tom: (Asking for clarification of the problem) It’s as if we were to start with the same number, starting with one and ending with the same number, right?

24 Int: …Do you have a way to begin with?

25 Tom: to go through every possible number and check it.

The verbal signifiers "the same number" and “every possible number” signify a generalized number. The first narrative [21] is embedded in what Tom thinks is given in the problem. This narrative implies the question: Is it true that we get the same number we started with? [21]. The second narrative is implied in the procedure Tom suggests for solving the problem. Like Gil and Mika, Tom uses the verbal signifier “the number I thought of” as a number which can potentially be any number, and therefore a generalized number.

As he continues to engage with the task, Tom formulates a non-canonical equation and at a certain point, he reaches the equation $\frac{x}{7}+54=\frac{x}{7}$ and says: "No, I just don't understand... How can something plus 54, be equal to the same thing?". The verbal signifier "something" indicates $\frac{x}{7}$, which is used in the equation as some numerical quantity, that is, as a signifier of a generalized number. We conclude that the narratives in Tom’s discourse revolve around generalized numbers signified both verbally and symbolically.

Last, we present Alon’s discourse. Alon without uttering a word, reads the problem, writes an equation and solves it.

1 Alon: ((Alon reads the question silently and writes))

2 Alon:

3 Alon: The number is nine.

We can see Alon’s narrative about a generalized number, in turn [3] where he states that “the number”, which is the number he was looking for, is indeed nine. This is an explicit verbal narrative about a generalized number.

When requested to provide an explanation to his solution, Alon explains

5 Alon: I gave the number an unknown which is x,

Here we see a narrative about the same “number” which Alon previously recalled was “nine” and which is associated with the “number I thought of” in the problem text, yet is also symbolically mediated. Most of Alon’s explanation onwards is devoted to describing his procedure of solving the equation. It includes descriptions of each of the steps Alon performed on the various symbolic signifiers. For example, “now I did… seven x…to move it (the sign x on the right side of the equation [2]) to here (the left side)” is a description of moving the sign x from one side of the equation to the other. This description does not include any narrative about a generalized number. We conclude that in Alon’s discourse there are explicit narratives about both verbal and symbolic generalized numbers, although his narratives about manipulating the equation refer to empty signifiers.

To summarize the findings, we use the following table (Table 1):

Table 1: Narratives and Canonical Procedures in Students’ Discourse

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| student | Narrative about specific numbers  | Narrative about verbal generalized numbers | Narrative about symbolic generalized numbers | Canonical symbolically mediated procedure |
| Mika | + | +/– | – | – |
| Liat | + | – | – | – |
| Gil | + | + | – | – |
| Tom | + | + | + | – |
| Alon | + | + | + | + |

*Note:* + indicates explicit narrative, – indicates no narrative and +/– indicates implicit narrative. When both implicit and explicit narratives were found we marked +.

From Table 1 we can locate the five students on a continuum from those who authored no narratives about generalized numbers at all (Liat) to those who produced narratives about generalized numbers that were also connected to symbolic mediators (Tom, Alon). We note that the narratives in Alon’s discourse are not only about generalized numbers (verbal and symbolic) but also canonical. In short, the placement on a continuum according to objectification of generalized numbers is Alon, Tom, Gil, Mika, and finally Liat.

Discussion and Conclusions

Our goal in this study was to suggest a methodological tool for differentiating levels of objectification in students’ algebraic discourse, focusing on the objectification of generalized numbers, or variable/unknown. Applying this methodological tool to the discourse of five students, we found that it enabled us to place the students’ solutions on a continuum, even though in terms of school mathematics, most of these students’ solutions could not be considered as canonical in the algebraic discourse. In that sense, our analysis unveiled phenomena that might be at odds with common assessment practices. For example, in Liat’s discourse we found various symbolic mediators, yet these symbols never functioned as labels of generalized numbers or parts of stories about. Her algebraic discourse was therefore purely syntactic, or ritual (Baccaglini-Frank, 2021). In contrast, we saw that students’ discourse which, at first looked purely “arithmetic” (such as Mika’s), could include implied narratives about generalized numbers, which at the least enabled the student to translate the story told in the problem’s text, to a meaningful story about specific numbers. We thus conclude that despite some students not using any algebraic signifiers in their discourse, their participation may be more explorative than those who do use such signifiers, yet without any reference to generalized numbers.

We believe our suggested method has potential to open multiple future avenues for research, both in terms of developing additional parts of assessment and diagnostic tools, as well as assisting in characterizing students’ algebraic discourse as they engage in school-based tasks. This potential stems from our method’s alignment with a developmental theory of mathematical learning which does not relate just to beginning algebra. Rather, the theoretical tools of ritual-exploration and objectification can (and have been) applied to multiple other topics in mathematics. This method therefore strengthens the power of commognition to map students’ algebraic discourse in its various stages. In the future, it is expected to enable mapping students’ discourse along the ritual-explorative continuum in a way that will allow quantitative research that enables comparison of students' discourse over time and between students. This will enable combining micro-analytical lenses, which are necessary for understanding subtle changes in learning, along with broader comparisons that require quantitative tools.

Acknowledgements

The research was supported by the Israel Science Foundation, Grant No. 744/20.

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